Text Book on

Principles of Real Analysis

As per U.G.C. Syllabus for B.A./B.Sc(Honours)
in
Mathematics
Under all colleges affiliated to Dibrugarh and Gauhati University.

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First Edition

by

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Preface

The book has been designed for the student of B.A/B.Sc(Honors) classes of Indian university in accordance with the new unified syllabus of Mathematics recommend by the board constituted by the U.G.C. of India. Every effort has been made to make the presentation of subject with well graded solved example, Simple and easily accessible to an average students without sacrificing. This text book incorporates the latest syllabus of various universities such as Dibrugrah and Gauhati.

We have tried to keep the book free from printing error. We shall be grateful to the professor and students who will point out the error or any other valuable suggestion to improve the quality of book.

Authors

SYMBOLS AND THEIR MEANINGS

$\exists$	"there exists"
$\forall$	"for every" or "for all"
$\Rightarrow$	"implies"
$\Leftrightarrow$	"implies and is implied by" or "if and only if"
$\in$	"belongs to"
$\notin$	"does not belong to"
$\subseteq$	"is a subset of" or "is contained in"
$\supseteq$	"is a super-set of"
$\subset$	"is a proper subset of"
$\not R$	"is not R-related to"
$:$ or $\vert$	"such that"
$\cup$	"union"
$\cap$	"intersection"
$\varnothing$	"the null set"
N	"the set of natural numbers"
I	"set of integers"
Q	"set of rational numbers"
R	"set of real numbers"
C	"set of complex numbers"
I_+	"set of positive integers"
R_+	"set of positive real numbers"
Q_+	"set of positive rational numbers"
I_0	"set of non-zero integers"
Q_0	"set of non-zero rational numbers"
R_0	"set of non-zero real numbers"
C_0	"set of non-zero complex numbers"

CONTENTS

Chapter 1

SEQUENCES OF REAL NUMBER

For any set S, A sequence is a function with domain N and with range subset of S. It we take S = R than this function is called the real sequence.

In this chapter we shall deal with real sequence.

1.1. Definition :

If N is a set of Natural numbers and R is a set of real numbers then the function f whose domain N and whose range is subset of R is called the real sequence. Symbolically we can say that $f : N \to R$ is real sequence.

If $f : N \to R$ is sequence then f(n) associate with unique real number. This no. is called the n^{th} term of this sequence. We write f(n) in other notation as x_n . We write the q sequence symbolically as $<x_n>$, $\{x_n\}$ x_1, x_2 ..... are called the First, Second ....term. We write any sequence by defining the n^{th} terms, e.g. if we have a sequence $<x_n> = <1,2,3,....., n, ...\to$ then we can also write it as $<n>$. other way to write a sequence is said to be inductively (or recursively). In this way we write the first term of sequence and give the formula for $x_{n+1} (n \geq 1)$ in terms of x_n.

e.g. if we have a sequence $< 2n >$ then it is inductively written as

$x_1 = 2$, $x_{n+1} = x_{n+2}$

1.2. Range :

Set of all distinct terms of a sequence is said to be its range.

e.g. $<1, 1, 1, ...>$ is any sequence then its range set is $\{1\}$

1.3. Constant Sequence :

A sequence $<x_n>$ is said to be constant sequence if $x_n = K \;\forall\; n \in N$ where $K \to R$.

e.g. $<1, 1, ...>$ is a constant sequence.

1.4. Subsequence :

If $<x_n>$ is any sequence and $n_1 < n_2 < ..... <n_m < ....$ is strictly increasing sequence of positive integer. Then the sequence $< x_{m_n} >$ is said to be subsequence of $<x_n>$

e.g. Take a sequence $<1, 0, 1, 0, 1, 0,...>$ then $<1, 1, 1, ...>$ & $<0, 0, 0, ...>$ are two subsequences.

Another example $<1, 1, 1, ...>$ & $<-1, -1, -1....>$ are subsequence of $<1, -1, 1, -1, ...>$

1.5. Bounded and Unbounded sequence :

Let $<x_n>$ is any sequence. This sequence is bounded above if $\exists$ a real no. M_1 s.t $x_n \leq M_1 \;\forall\; n \in N$

This sequence is said to be bounded below if $\exists$ a real no. m_1s.t. $m_1 \leq x_n \;\forall\; n \in N$

A sequence $<x_n>$ is called bounded sequence if It is both ie. it is bounded above as well as bounded below.

It the sequence $<x_n>$ is not bounded then it is called unbounded sequence.

EXAMPLE

(i) $\langle x_n = \dfrac{1}{n} \rangle$ is bounded sequence since $0 < x_n < 1 \;\forall\; n \in N$

(ii) $<(-1)^n>$ is bounded since $-1 \leq x_n \leq 1 \;\forall\; n \in N$

(iii) Other bounded sequences are

$$\langle \frac{n}{n+1} \rangle, \langle \frac{(-1)^n}{x} \rangle, \langle 1 - (-1)^n \rangle$$

(iv) The sequence $< -n >$ is not bounded actually it is bounded above because $x_n < 1 \;\forall\; n \in N$. We can say it is unbounded below because we can not find any real no. m_1 st. $m_1 \leq x_n \;\forall\; n \in N$

(v) The sequence $< n >$ is not bounded. It is unbounded or we can say it is unbounded above.

(vi) $<(-1)^n n>$ is unbounded. It is unbounded below as well as unbounded above or we can say it is neither bounded below nor bounded above.

1.6. Supremum and Infimum of the sequence :

The least no. M of the set of upper bound of a sequence $<x_n>$ if exist, is said to be supermum or least upper bound of the sequence $<x_n>$.

The greatest no. m of the set of lower bound of a sequence $<x_n>$, if exist, is said to be Infimum or greatest lower bound.

Theorem 1:

A sequence $<x_n>$ is bounded if an only if $\exists$ a natural no. m, $l \in R$ and $K > 0$ s.t. $|x_n - l| < K \;\forall\; n \geq m$.

Proof :

It is given $< x_n >$ is bounded. So $\exists$ two real no. m_1 and M_1 s.t.

$m_1 < x_n < M_1 \;\forall\; n \in N$

$$m_1 - \frac{m_1 + M_1}{2} \langle x_n - \frac{m_1 + M_1}{2} \langle M_1 - \frac{m_1 + M_1}{2} \forall n \in N$$

$$\text{or, } \frac{m_1 - M_1}{2} \langle x_n - \frac{m_1 + M_1}{2} \langle \frac{M_1 - m_1}{2} \forall n \in N$$

$$If \frac{M_1 - m_1}{2} = k(\text{say}) \ \& \ \frac{M_1 + m_1}{2} = \ell(\text{say}) \text{ then } -k < x_m - l < k$$

$\Rightarrow |x_n - l| < k \ \forall n \in N$

$\Rightarrow |x_n - l| < k \ \forall n \in m$ Where $m = 1 \in N$, $l \in R$, $k > 0$

Conversely, if $l \in R$, $k > 0$, $m \in N$ s.t.

$|x_n - l| < k \ \forall \ n \geq m$.

$\Rightarrow l - k < x_n < l + k \ \forall \ n \geq m$.

Let $m_1 = \min\{x_1, x_2, \ldots\ldots x_{m-1}, l - k\}$

$M_1 = \max\{x_1, x_2 \ldots., x_{m-1}, l + k\}$

then $m_1 \leq x_n \leq M_1 \ \forall n \in N$

so $< x_n >$ is bounded.

1.7. Limit :

Definition :

Let $< x_n >$ be a sequence in R. Then $< x_n >$ is converge to $l \in R$ or l is limit of $< x_n >$ if to each $\in <0$ $\exists$ a + ve integer m (depending on $\in$) s.t. $|x_n - l| < \in \forall n \geq m$ if a sequence $<x_n>$ has a limit l then we say that sequence $<x_n>$ is convergent and converges to l.

If a sequence $<x_n>$ has a limit l then we write $\lim\limits_{n\to\infty} \langle x_n \rangle = \ell$ or $\lim\limits_{n\to\infty} x_n = \ell$

Theorem 2:

Every convergent sequence is bounded but converse is not true.

Proof :

Let $<x_n>$ be any sequence converging to l

$l$im $x_n = l$. By Definition

take $\in = 1$, $\exists$ a + ve integer m s.t.

$|x_n - l| < 1 \ \forall n \geq m$.

$l - 1 < x_n < l + 1 \ \forall n \geq m$

If, $M_1 = \min\{x_1, x_2 \ldots.., x_{m-1}, l - 1\}$ and

$M_2 = \max\{x_1, x_2 \ldots., x_{m-1}, l + 1\}$

Then $M_1 \leq x_n \leq M_2 \ \forall \ n \in N$

So, $<x_n$ is bounded.

Converse of the above the orem is not true for Example take

$<(-1)^n>$. This sequence is bounded since $-1 \leq x_n \leq 1 \ \forall \ n \in N$

$x_n = (-1)^n$

$\mathit{lim}\ x_{2n} = \mathit{lim}\ (-1)^{2n} = \mathit{lim}\ 1 = 1$

$\mathit{lim}\ x_{2n+1} = \mathit{lim}\ (-1)^{2n+1} = \mathit{lim}\ (-1) = -1$

$\mathit{lim}\ x_n$ = does not exist. So $< x_n >$ does not convergent $<(-1)^n>$ oscillates finitely.

Theorem 3: Every convergent $< x_n >$ has unique limit.

Proof :

Let $<x_n>$ be a sequence converges to l_1 & l_2 then by definition we have to each $\in > 0$, $\exists$ a + ve integer m_1 s.t.

$| x_n - l_1| < \in/2 \ \forall \ n \geq m_1$ and $\exists \ m_1$ ---(1)

and $\exists \ m_2$ s.t.

$| x_n - l_2| < \in/2 \ \forall \ n \geq m_1$ and $\exists \ m_2$ ---(2)

If $m = \max \{m_1, m_2\}$

Then for $n \leq m$ we use the triangle inequality to get

$| l_1 - l_2| = |l_1 - x_n + x_n - l_2|$

$\leq |x_n - l_1| + |x_n - l_2|$

$$\langle \frac{\in}{2} + \frac{\in}{2} = \in$$

From (1) & (2)

$\Rightarrow | l_1 - l_2| < \in$

Since $\in$ is arbitrary so we conclude $l_1 = l_2$

1.8. Divergent Sequence :

A sequence $< x_n>$ is diverge to $+ \infty$ if for a given $M \in R^+$, however large, $\exists$ a + ve integer m s.t. $x_n > M \ \forall \ n \geq m$.

If $< x_n>$ diverge to $+ \infty$ then we write $\mathit{lim}\ x_n = \infty$ or $x_n \to \infty$ as $n \to \infty$ Sequence $<x_n>$ is diverge to $-\infty$ if for a given $M \in R^-$, However large, $\exists$ a + ve integer m s.t. $x_n < M \ \forall \ n \geq m$.

If $< x_n >$ diverge to $- \infty$ then we write $\mathit{lim}\ x_n = -\infty$ or $x_n \to -\infty$ as $n \to \infty$

A sequence is divergent if it is either diverge to $+ \infty$ or $- \infty$

1.9. Oscillatory sequence :

A sequence $< x_n >$ which is neither convergent nor divergent is said to be oscillatory sequence an oscillatory sequence is oscillate finitely if it is bounded. If it is unbounded then it is oscillate infinity.

IMPORTANT POINTS

1. Sequence $\langle \frac{1}{n} \rangle$ is convergent.
2. Sequence $< r^n >$ is convergent, $| r | < 1$
3. Sequence $\langle \frac{(-1)^n}{n} \rangle$ is convergent.
4. Sequence $< n >$, $< n^2 >$ are diverges to + ¥
5. Sequence $< -n >$, $< -n^2 >$ are diverges to – ¥
6. $< (-1)^n >$ oscillates finitely.
7. $< (-1)^n.n >$ oscillates infinitely.

Theorem 1 : Let $< x_n >$ and $< y_n >$ are two sequences if $\lim x_n = l_1$ and $\lim y_n = l_2$ then $\lim (x_n \pm y_n) = l_1 \pm l_2$

Proof :

Since $\lim x_n = l_1$ so to each $\in > 0$, (a + ve integer m_1 s.t.

$$\left|x_n - l_1\right| < \frac{\in}{2} \forall n \geq m_1 \qquad \text{---(1)}$$

Again $\lim y_n = l_2$ so to each $\in > 0$ $\exists$

a + ve integer m_2 s.t. $\left|y_n - \ell_2\right| \langle \frac{\in}{2} \forall n \geq m_2$ ---(2)

Now if $m = \max\{m_1, m_2\}$

Then,

$$\left.\begin{array}{l} \left|x_n - \ell_1\right| < \frac{\in}{2} \forall n \geq m \\ \& \quad \left|y_n - \ell_2\right| < \frac{\in}{2} \forall n \geq m \end{array}\right] \qquad \text{---(3)}$$

$|(x_n + y_n) - (l_1 + l_2)| = |x_n - l_1 + y_n - l_2|$

$\leq |x_n - l_1| + |y_n - l_2|$

$\langle \frac{\in}{2} + \frac{\in}{2} \; \forall \, n \geq m$

$= \in$ From (3)

So the sequence $< x_n + y_n >$ is convergent and $\lim(x_n + y_n) = l_1 + l_2 = \lim x_n + \lim y_n$

Similarly we can easily show that

$\lim(x_n - y_n) = l_1 - l_2 = \lim x_n - \lim y_n$

Theorem 2: If $\lim x_n = l_1$ and

$lim\ y_n = l_2$ then $\lim(x_n \cdot y_n) = l_1 l_2 = \lim x_n \cdot \lim y_n$

Proof :

We take $| x_n \cdot y_n - l_1 l_2 | = | x_n y_n - x_n l_2 + x_n l_2 - l_1 l_2|$

$= | x_n (y_n - l_2) + l_2 (x_n - l_1)| \leq | x_n (y_n - l_2) | \, | l_2 (x_n - l_1)|$

$= | x_n | \, | y_n - l_2| + | l_2| \, |x_n - l_2|$ ---(1)

Since $< x_n >$ is converges to l_1 and we know Every convergent sequence is bounded. So $\exists$ a no.

$M > 0$ s.t. $| x_n| \leq M \; \forall$ n.

If $k = \text{Sup.}\{M_1 \, |l_2|\}$

So from (1) $|x_n . y_n - l_1 l_2 \leq k \, \{|y_n - l_2| + |x_n - l_1|\}$ ---(2)

Since $< x_n >$ converges to l_1 so to each $\in > 0 \rightarrow$ a + ve integer M_1 s.t. $|x_n - \ell_1| \langle \frac{\in}{2k} \; \forall \, n \geq M_1$. Since $< y_n >$ converge to l_2 so to each $\exists > 0$ (a + ve integer M_2

s.t. $|y_n - \ell_2| \langle \frac{\in}{2k} \; \forall \, n \geq M_2$

If $M^1 = \text{Sup.}\{M_1 , M_2\}$

$$\text{So,} \left. \begin{array}{l} |x_n - \ell_1| \langle \frac{\in}{2k} \; \forall \, n \geq M' \\ |y_n - \ell_2| \langle \frac{\in}{2k} \; \forall \, n \geq M' \end{array} \right] \quad \text{---(3)}$$

From (2) & (3)

$$|x_n \, y_n - \ell_1 \ell_2| \langle k \left[\frac{\in}{2k} + \frac{\in}{2k} \right] \forall \, n \geq M'$$

$< \in \quad \forall\ n \geq M^1$

Hence $\lim x_n\, y_n = l_1\, l_2 = \lim x_n . \lim y_n$

Theorem 3 : If $\lim x_n = l\ (\neq 0)$ and $x_n \neq 0\ \ \forall$ n then $\ell im \dfrac{1}{x_n} = \dfrac{1}{\ell}$

Proof :

$\because\ l \neq 0$ (a real no. M > 0 and a + ve integer m' s.t.

$M < |\, x_n \,| \quad \forall\ n \geq m'$ ---(1)

Now it is given that $< x_n >$ converges to l so to each $\in > 0\ \exists$ a + ve integer

m^* s.t. $|\, x_n - l \,| < M\, |\, l \,| \in \quad \forall\ n \geq m^*$ ---(2)

If $m = \max\,(m^1, m^*)$ then

$|\, x_n \,| < M \quad \forall\ n \geq m$

& $|\, x_n - l \,| < M\, |\, l \,| \in \quad \forall\ n \geq m$

i.e. (1) & (2) hold for $n \geq m$.

Now, $\left|\dfrac{1}{x_n} - \dfrac{1}{\ell}\right| = \left|\dfrac{x_n - \ell}{|x_n||\ell|}\right| < \dfrac{M\,|\ell|}{M\,|\ell|} \in \forall n \geq m$ From (3)

$$\Rightarrow \left|\frac{1}{x_n} - \frac{1}{\ell}\right| < \in \forall n \geq m$$

$\Rightarrow \ell \text{im} \dfrac{1}{x_n} = \dfrac{1}{\ell}$ ie. $\left\langle \dfrac{1}{x_n} \right\rangle$ converges to $\dfrac{1}{\ell}$

Theorem 4 : If $lim\ x_n = l_1$ & $lim\ y_n = l_2$ where $l_2 \neq 0$ and $y_n \neq 0 \quad \forall\ n \in N$. Then.

$$\ell \text{im} \left(\frac{x_n}{y_n}\right) = \frac{\ell_1}{\ell_2}$$

Proof :

From the theorem 4 we have if $\lim y_n = l_2$ then $\ell \text{im} \dfrac{1}{y_n} = \dfrac{1}{\ell_2}$ and from the theorem 2 lim $(x_n - y_n) = \lim x_n . \lim y_n = l_1 . l_2$

So, $\lim\left(\frac{x_n}{y_n}\right) = \lim\left(x_n.\frac{1}{y_n}\right)$

$= \ell\text{im}\, x_n \cdot \ell im \frac{1}{y_n}$

$= \ell_1 \cdot \frac{1}{\ell_2} = \frac{\ell_1}{\ell_2}$

Theorem 5 : If the sequence $< x_n >$ converges to l then the sequence $<| x_n |>$ converges to $| \ell |$.

Proof :

It is given that sequence $< x_n >$ converges to l so we have to each $\in > 0 \rightarrow$ a + ve integer m (depending on $\in$) s.t.

$|x_n - l | < \in \ \forall\ n \geq m$ ---(1)

$\because | \, | x_n | - | l | \leq | x_n - l |$

So, $| \, | x_n | - | l | \, | < \in \ \forall\ n \geq m$ From (1)

$\Rightarrow < | x_n | >$ Converges to l.

Theorem 6 : If $< x_n >$ is a convergent sequence s.t. $x_n \geq 0 \ \forall\ n \in N$ and $\lim x_n = l$ then $l \geq 0$

Proof :

Let $l < 0$.

$\because \lim x_n = l$ so to each $\in > 0 \ \exists$ a + ve integer m s.t.

$| x_n - l | < \in \ \forall\ n \geq m$

ie. $l - \in < x_n < l + \in \ \forall\ n \geq m$.

We choose $\in = - l$

So For $\in = - l > 0 \ \exists$ a + ve integer m' s.t.

$l + l < x_n < l - l \ \forall\ n \geq m'$

$\Rightarrow 2l < x_n < 0 \ \forall\ n \geq m'$

or $x_n < 0 \ \forall\ n \geq m'$

Which is contradiction because $x_n \geq 0 \ \forall\ n \in N$.

So our assumption is wrong.

Hence $l \geq 0$.

Theorem 7 : If $<x_n>$ and $<y_n>$ are two sequences s.t. $x_n \leq y_n \ \forall\ n \in N$ and $\lim x_n = l_1$,

$\lim y_n = l_2$ then $l_1 \le l_2$

Proof :

Consider $z_n = y_n - x_n$

$z_n \ge 0 \quad \forall\, n \in N \qquad [\because x_n < y_n]$

Then from the theorem (6) we have

$\lim z_n \ge 0$

i.e. $\lim(y_n - x_n) \ge 0$

$\Rightarrow \lim y_n - \lim x_n \ge 0$

$\Rightarrow \lim y_n \ge \lim x_n$

$\Rightarrow l_1 \le l_2$

Theorem 8 : (Sandwich theorem) Let $<h_n>$, $<g_n>$ and $<t_n>$ are three sequences s.t. $h_n \le g_n \le t_n \ \forall\, n \in N$ and $\lim h_n = l = \lim t_n$ then $\lim g_n = l$.

Proof :

Given $h_n \le g_n \le t_n$ ---(1)

Since $\lim h_n = \lim t_n = l$, so for given

$\in > 0\ \exists$ + ve integers m', m'' s.t.

$| h_n - l | < \in \quad \forall\, n \ge m_1 \qquad (2)$

& $| t_n - l | < \in \quad \forall\, n \ge m_2 \qquad (3)$

if $m = \max\{m_1, m_2\}$ then (2) & (3) holds for $n \ge m$

From (1) , (2) , (3)

$l - \in < h_n \le y_n \le t_n < l + \in \qquad \forall\, n \le m$

$\Rightarrow l - \in < g_n < l + \in \qquad \forall\, n \ge m$

$\Rightarrow | g_n - l | < \in \qquad \forall\, n \ge m$

So, $\lim g_n = l$

Theorem 9 : Any subsequence $\langle x_{n_k} \rangle$ of a sequence $\langle x_n \rangle$ which diverge to infinity is also diverge to infinity.

Proof :

Given $<x_n>$ diverge to ∞. So By definition for a given $M \in R^+\ \exists$ a + ve integer m s.t

$x_n > M \quad \forall\, n \ge m$

Since $\langle x_{n_k} \rangle$ is subsequence of $\langle x_n \rangle$ so $< n_1, n_2 ...., n_k, ... >$ is strictly increasing sequence of Natural numbers.

We have $n_1 \geq 1$. By induction we can easily show that $n_k \geq k$

if $k \geq m$, then

$n_k \geq k \geq m \Rightarrow n_k \geq m$

So, $x_{n_k} > M \quad \forall\, n \geq m$

$\Rightarrow \langle x_{n_k} \rangle$ diverge to infinity.

Theorem 10 : Let $<x_n>$ & $<y_n>$ are two sequences and both are diverges to infinity. The sequences $<x_n + y_n>$ and $<x_n\, y_n>$ diverges to infinity.

Proof :

Since $<x_n>$ and $<y_n>$ both are diverge to infinity so for a given $M_1 \in R^+$, $\exists$ a + ve integer m_1 s.t.

$x_n > M_1 \quad \forall\, n \geq m_1$

and for a given $M_2 \in R^+ \exists$ a + ve integer m_2 s.t. $y_n > M_2 \quad \forall\, n \geq m_2$

If $m = \max(m_1, m_2)$ then

$(x_n + y_n) > M_1 + M_2 = M$ (say)

& $x_n\, y_n > M_1\, M_2 = M^1$ (say)

$\Rightarrow < x_n + y_n >$ & $< x_n\, y_n >$ both are diverge to infinity.

Theorem 11: Let $<x_n>$ is sequence s.t. $x_n > 0 \ \forall\, n \geq N$. Then x_n diverge to infinity iff $\left\langle \frac{1}{x_n} \right\rangle$ converge to zero.

Proof :

Given $\lim x_n = \infty$. so for $\frac{1}{\in} > 0$, $\exists$ a + ve integer m s.t. $x_n > \frac{1}{\in} \quad \forall\, n \geq m$

$$\Rightarrow \frac{1}{x_n} < \in$$

$$\Rightarrow \frac{1}{x_n} - 0 < \in$$

$$\Rightarrow \left|\frac{1}{x_n} - 0\right| < \in \forall n \geq m$$

$$\Rightarrow \lim \frac{1}{x_n} = 0$$

ie. $\left\langle \frac{1}{x_n} \right\rangle$ converges to zero.

IInd Part : Let MÎR$^+$ any arbitrary number. Since $\lim \frac{1}{x_n} = 0$ **so for** $\frac{1}{M} \rangle 0 \exists$ **a + ve Integer m s.t.** $\left|\frac{1}{x_n} - 0\right| \langle \frac{1}{M} \quad \forall n \geq m$

$$\Rightarrow \frac{1}{x_n} \langle \frac{1}{M} \quad x_n \rangle 0 \quad \forall n \in N$$

$\Rightarrow x_n > M \quad \forall\, n \geq m$

So $\lim x_n = \infty$

$<x_n>$ diverges to infinity.

Theorem 12 : If $\lim x_n = \infty$ then $<x_n>$ is bounded below but not bounded above, if $\lim x_n = -\infty$ then $<x_n>$ is bounded above but not bounded below.

Proof :

Since $< x_n>$ diverges to ∞, so by definition, for a given $M \in R^+$ $\exists$ a + ve integer m s.t.

$M < x_n \quad \forall\, n \geq m$

There are finitely many terms in $<x_n>$ which are $\leq$ M.

Sp, $<x_n>$ is unbounded above

Consider M = 1

$1 < x_n \quad \forall\, n \geq m$

Let $M_1 = \min\{x_1, x_2, ...x_{m-1}, 1\}$

Then $M_1 \leq x_n \quad \forall\, n \in m$

So, $<x_n>$ is bounded below similarly we can prove the second part easily.

EXAMPLE

1. Prove that the sequence $\left\langle \frac{1}{n} \right\rangle$ has the limit 0.

Sol. : Let $\in > 0$ be given

$\left| \frac{1}{n} - 0 \right| = \left| \frac{1}{n} \right| = \frac{1}{n} < \in$

When $\frac{1}{\in} \langle$ n

Now we select + ve integer $m \rangle \frac{1}{\in}$

then $\left| \frac{1}{n} - 0 \right| \langle \in \quad \forall n \geq m$

So $\left\langle \frac{1}{n} \right\rangle$ has the limit Zero.

2. By the use of Definition, show that the sequence $\left\langle \frac{3n-1}{4n+5} \right\rangle$ **converging to** $\frac{3}{4}$**.**

Sol. :

Let $\in > 0$ be given,

$\left| \frac{3n - 1}{4n + 5} - \frac{3}{4} \right| = \left| \frac{12n - 4 - 12n - 15}{4(4n + 5)} \right|$

$= \left| \frac{-19}{4(4n + 5)} \right|$

$= \left| \frac{19}{4(4n + 5)} \right| = \frac{19}{4(4x + 5)} \langle \frac{19}{n} \langle \in$

When $\frac{19}{n} \langle \in$

if we select $m \rangle \frac{19}{\in}$ then

$$\left|\frac{3n-1}{4n+5}-\frac{3}{4}\right| < \in \quad \forall n \geq m$$

So the sequence $\left\langle \frac{3n-1}{4n+5} \right\rangle$ Converges to $\frac{3}{4}$

3. Show that $\left\langle \frac{n+1}{n} \right\rangle$ converges to 1. (by use of definition)

Sol. :

Let $\in > 0$ be given.

$$\left|\frac{n+1}{n} - 1\right| = \left|\frac{n+1-n}{n}\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \langle \in$$

When $\frac{1}{\in} \langle n$

If we select $m \rangle \frac{1}{\in}$ then $\left|\frac{n+1}{n} - 1\right| \langle \in \quad \forall n \geq m$

So the sequence $\left\langle \frac{n+1}{n} \right\rangle$ converging to 1.

4. By use of Definition show that the sequence < n > diverging to ∞.

Sol. :

Let $M \in R^+$ is given.

Since if a and b any two + ve real numbers then $\exists$ a + ve integer n_1 s.t. $n_1 a > b$ (Archimedean property).

If we take a = 1 then $n_1 > b$.

Now take $b = m$ & $n_1 = m$ we have $m > M$

For Every $n \geq m$ and $m > M$, we have

$n \geq m > M$ ie. $n > M$

So, $n > M \quad \forall n \geq m$

$\Rightarrow x_n > M \quad \forall n \geq m$

Hence, $< x_n = n >$ diverge to ∞.

5. Prove that (by definition) the sequence $\left\langle \log \frac{1}{n} \right\rangle$ diverge to $-\infty$.

Sol. : Let $M \in R^+$ given

$$\log \frac{1}{n} < -M$$

If $(\log 1 - \log n) = -\log n < -M$

If $\log n > M$

If $n > e^M$

We select a + ve integer $m > e^M$ then $\log \frac{1}{n} < -M \quad \forall\, n \geq m$.

So the sequence $\left\langle \log \frac{1}{n} \right\rangle$ diverging to $-\infty$.

6. Is the sequence $\left\langle \frac{n}{n+1} \right\rangle$ is bounded ?

Sol. :

$$x_n = \frac{n}{n+1}$$

$$\lim x_n = \lim \frac{n}{n+1} = 1$$

$< x_n >$ converging to 1 ie. $\left\langle x_n = \frac{n}{n+1} \right\rangle$ is convergent. Since Every convergent sequence is bounded. So given sequence $\left\langle \frac{n}{n+1} \right\rangle$ is bounded.

7. Prove that the sequence $<(-1)^n\, n>$ oscillate infinity ?

Sol. :

Given sequence is $< (-1)^n\, n>$ ie.

$< \ldots\ldots -5, -3, -1, 2, 4, 6, \ldots\ldots>$

Since we can not find a + ve real no. M s.t. $|x_n| \leq M \quad \forall\, n \in N$ or the range set of this sequence is unbounded so this sequence is not bounded i.e. it is unbounded sequence.

Now, $\lim x_{2n} = \lim [(-1)^{2n} . 2n] = \infty$

$\lim x_{2n+1} = \lim [(-1)^{2n+1} . 2(n+1)] = -\infty$

The sequence is not divergent.

Hence the given sequence $<(-1)^n n>$ oscillates infinitely.

1.10. Monotonic Sequence :

A sequence $< x_n >$ is called monotonic if it is either monotonically increasing or monotonically decreasing sequence. $< x_n >$ is called monotonically increasing sequence if $x_n \leq x_{n+1} \quad \forall n \in N$

$<x_n>$ is called monotonically decreasing sequence if $x_n \leq x_{n+1} \quad \forall n \in N$.

If $x_n < x_{n+1} \quad \forall n \in N$ then $<x_n>$ is called strictly monotonically increasing and if $x_n > x_{n+1}$ $\forall n \in N$ then it is called strictly monotonically decreasing sequence.

If $<x_n>$ is strictly monotonically increasing or strictly monotonically decreasing then it is called strictly monotonic sequence.

IMPORTANT POINT

1. Sequence < 1, 2, 2, 3, 3, 3, ... > is monotonically increasing.
2. Sequence < 1, 2, 3, ...> is strictly monotonically increasing sequence.
3. $\left\langle -\frac{1}{n} \right\rangle$ is strictly increasing sequence.
4. $<-2n>$ is strictly decreasing sequence.
5. <1, 0, 1, 0, 1, 0, ... > is not monotonic.

Theorem 1 :

Monotone Convergence theorem : Every Monotonic Sequence is Convergent iff it is bounded. Again if $< \alpha_n >$ is bounded and monotonically increasing then $\lim \alpha_n = l_1$ where $l_1 = \text{Sup}\{\alpha_n | n \in N\}$ and if $<t_n>$ is bounded and monotonically decreasing then $\lim t_n = l_2$ where $\ell_z = \text{Inf.}\{t_n\} n \in N$

Proof :

Let $< \alpha_n >$ be monotonic and convergent. we have already proved in a theorem that Every convergent sequence is bounded.

If $< \alpha_n >$ is bounded monotonic sequence then it is either monotonically increasing or monotonically decreasing sequence. If $< \alpha_n >$ is bounded and monotonically increasing now it is given $< \alpha_n >$ is bounded so $\exists$ a real no. k_1 s.t.

$\alpha_n \le k_1 \quad \forall\ n \in N$

By completeness property l_1 = Sup $\{\alpha_n | n\in N\} \in R$ Now $\in > 0$ be given. $l_1 - \in$ is not upper bound of $\{\alpha_n | n\in N\}$ So $\exists$ an elements α_m of set $\{\alpha_n | n\in N\}$ s.t. $l_1 - \in < \alpha_m$. Since $< \alpha_n >$ is monotonically increasing.

So, $\alpha_m \le \alpha_n$ when $n \ge m$

Thus we have $l_1 - \in < \alpha_m \le \alpha_n \le l_1 < l_1 + \in \quad \forall\ n \ge m$.

$\Rightarrow\ l_1 - \in < \alpha_n < l_1 + \in \quad \forall\ n \ge m$

$\Rightarrow |\alpha_n - l_1| < \in \quad \forall\ n \ge m$

$\Rightarrow \lim \alpha_n = l_1 = \text{Sup } \{\alpha_n \mid n\in N\}$

ie. α_n converges to l_1

let $< t_n >$ is bounded and monotonically decreasing then $<-t_n>$ is bounded and monotonically increasing.

Similarly from above we can show that

$\lim(-t_n) = \text{Sup } \{-t_n \mid n\in N\}$

We know if x is bounded and non-empty set in R and if a<0,

$ax = \{ax \mid x\in x\}$ then

inf (ax) = a Sup x,

Sup (ax) = a Inf x

So $\lim (-t_n) = -\text{ Inf } \{t_n \mid n\in N\}$

$\Rightarrow\ \lim t_n = \text{Inf } \{t_n\}\ n \in N = l_2$

$\Rightarrow < t_n >$ Converging to l_2

Corollary 1: If $< x_n >$ is monotonically increasing and unbounded above then it is diverge to ∞.

Proof :

Given $< x_n >$ is monotonically increasing and unbounded above sequence.

Suppose $M\in R$, However larger.

$\because < x_n >$ is unbounded above and monotonically increasing so $\exists$ a $m \in I^+$ s.t.

$x_m > M$ ---(1)

and $x_n \ge x_m \quad \forall\ n \ge m$ ---(2)

From (1) & (2)

$\Rightarrow x_n > M \quad \forall\ n \ge m$

$\Rightarrow < x_n >$ diverge to ∞.

Corollary 2 : Every unbounded below monotonically decreasing sequence diverge to $-\infty$

Proof :

Proof is easy for reader.

Corollary 3 : A sequence which is monotonic either convergent or divergent.

EXAMPLE

1. Show that the sequence $\left\langle \frac{3n+7}{4n+8} \right\rangle$ is monotonic. Is it monotonically increasing or decreasing ?

Sol. :

$$x_n = \left\langle \frac{3n+7}{4n+8} \right\rangle$$

$$x_{n+1} = \frac{3(n+1)+7}{4n+8} = \frac{3n+10}{4n+12}$$

$$x_{n+1} - x_n = \frac{3n+10}{4n+12} - \frac{3n+7}{4n+8}$$

$$= \frac{3n + 10}{4(n+3)} - \frac{3n + 7}{4(n+2)}$$

$$= \frac{1}{(4n + 12)(n + 2)} < 0 \ \forall\ n$$

So, $x_{n+1} - x_n < 0$

$\Rightarrow x_{n+1} < x_n \ \forall\ n$

So the given sequence is monotonically decreasing.

Hence $\left\langle \frac{3n+7}{4n+8} \right\rangle$ is monotonic.

2. Show that the sequence $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}}$, $n \geq 2$ is monotonic.

Sol. :

$x_1 = 1$

$$x_n = \sqrt{2 + x_1} = \sqrt{2 + 1} = \sqrt{3} > x_1 = 1$$

Now, let $x_n > x_{n-1}$

$\Rightarrow 2 + x_n > 2 + x_{n-1}$

$\Rightarrow \sqrt{2 + x_n} > \sqrt{2 + x_{n-1}}$

$\Rightarrow x_{n+1} > x_n$

So we have $x_{n+1} > x_n \quad \forall\, n \in N$.

Thus we have the given sequence is monotonically increasing. Hence given sequence is monotonic.

3. Show that the sequence $\left\langle \left(1+\frac{1}{n}\right)^n \right\rangle$ is convergent.

Sol. :

Given sequence is $\left\langle \left(1+\frac{1}{n}\right)^n \right\rangle$ take $x_n = \left(1+\frac{1}{n}\right)^n$

$$\text{So, } x_n = 1 + \frac{n}{n} + \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 + \ldots\ldots\ldots\ldots$$

$$+ \text{----} \frac{n(n-1)\ldots\ldots.3\ .\ 2.\ 1}{\angle n}\left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \frac{1}{\angle 2}\left(1 - \frac{1}{n}\right) + \text{----} + \frac{1}{\angle n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$---\left(1-\frac{n-1}{n}\right)$$

$$x_{n+1} = 1 + 1 + \frac{1}{\angle 2}\left(1 - \frac{1}{n+1}\right) + \text{----}$$

$$+ \frac{1}{\angle n+1}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)---\left(1-\frac{n}{n+1}\right)$$

Here $x_{n+1} > x_n$ since term in x_{n+1} is greater than or equal to the corresponding term in x_n and x_{n+1} has one more term than x_n which is + ve

So, $x_{n+1} > x_n \quad \forall\, n \in N$

$\Rightarrow$ Given series is monotonically increasing. Given sequence is bounded. We have

$$x_n = 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{\angle 2} + ---- +$$

$$\frac{1}{\angle n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)---\left(1-\frac{n-1}{n}\right)$$

$$x_n \langle\ 1 + 1 + \frac{1}{\angle 2} + \frac{1}{\angle 3} + ---- + \frac{1}{\angle n-1} + \frac{1}{\angle n}$$

$\langle\ 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ---- + \frac{1}{2^{n-1}}$ which is G.P. after first term.

$$= 1 + \frac{1 \cdot \left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}$$

$$= 1 + 2\left(1 - \frac{1}{2^n}\right) \langle\ 1 + 2 = 3$$

$x_n < 3 \quad \forall\ n \in N$

So, $< x_n >$ is bounded above.

Now $< x_n >$ is monotonically increasing and bounded above.

Hence the given sequence $\left\langle x_n = \left(1 + \frac{1}{n}\right)^n \right\rangle$ is convergent.

$\lim\left(1 + \frac{1}{n}\right)^n = e$ which lie between 2 and 3.

4. Show that the sequence $x_1 = \sqrt{2}, x_{n+1} = \sqrt{2x_n}$ Converges to 2 ?

Sol. :

$$x_1 = \sqrt{2}$$

Put $n = 1$ in $x_{n+1} = \sqrt{2x_n}$

We have $x_2 = \sqrt{2\,x_1} = \sqrt{2\sqrt{2}} \rangle \sqrt{2} = x_1$

Now suppose $x_n \langle x_{n+1}$

$\Rightarrow 2x_n < 2\,x_{n+1}$

$\Rightarrow \sqrt{2x_n} \langle \sqrt{2x_{n+1}}$

$\Rightarrow x_{n+1} < x_{n+2} \quad \forall\, n \in N$

So given sequence is monotonically increasing.

Since given sequence is monotonically increasing and $x_1 = \sqrt{2}$. So it is bounded below .

$x_1 = \sqrt{2} \langle 2$

Let $x_n < 2$

$\Rightarrow \sqrt{2\,x_n} \langle 2$

$\Rightarrow x_{n+1} < 2 \quad \forall\, n \in N$

So sequence is bounded above by 2 sequence is monotonically increasing and bounded.

Hence by Monotone Convergence theorem it is convergent.

Let $\lim x_n = l$ & $\lim x_{n+1} = l$

We have $x_{n+1} = \sqrt{2\,x_n}$

$x_{n+1}^2 = 2\,x_n$

$\lim x_{n+1}^2 = 2 \lim x_n$

$l^2 = 2l$

$\Rightarrow l^2 - 2l = 0$

$\Rightarrow l\,(l - 2) = 0$

$\Rightarrow l = 0, 2$

$l \neq 0$

$x_1 \leq x_n \;\forall\; n$

$\sqrt{2} \leq x_n$

$\lim x_n \geq \sqrt{2}$

So $l \neq 0$ therefore $l = 2$

Hence the sequence converging to 2.

5. Show that the sequence $x_1 = 1,\ x_{n+1} = \dfrac{4 + 3x_n}{3 + 2x_n}$, $n \geq 1$ is convergent ? What is the limit of this sequence ?

Sol. :

We have to show that given sequence is monotonically increasing and bounded above to show monotonically increasing we use mathematical induction method.

$$x_1 = 1,\ x_{n+1} = \frac{4 + 3x_n}{3 + 2x_n},\ n \geq 1$$

$x_1 = 1$

$$x_2 = \frac{4 + 3 \cdot 1}{3 + 2 \cdot 1} = \frac{7}{5} > x_1 = 1$$

Let $x_n < x_{n+1}$

$$x_{x+2} - x_{n+1} = \frac{4+3x_{n+1}}{3 + 2x_{n+1}} - \frac{4 + 3x_n}{3 + 2x_n}$$

$$12+8x_n+9x_{n+1}+6x_{n+1}x_n$$

$$= \frac{-12+8x_{n+1}-9x_n-6x_{n+1}x_n}{(3+2x_{n+1})(3+2x_n)}$$

$$= \frac{x_{n+1} - x_n}{(3+2x_{n+1})(3 + 2x_n)} \rangle\, 0$$

because $x_{n+1} > x_n$

So $x_{n+1} > x_{n+1}$

By Mathematical Induction sequence is monotonically increasing.

Now we show that the given sequence is bounded

$$x_1 = 1 \langle\, \frac{3}{2}$$

Let $x_n \,\langle\, \dfrac{3}{2}$

$$x_{n+1} = \frac{4 + 3x_n}{3 + 2x_n}$$

$$= \frac{3}{2} - \frac{1}{2(3 + 2x_n)}$$

$$\langle \frac{3}{2}$$ [$\because x_1 = 1$ and sequence is monotonically increasing so $x_n > 1$ so $\frac{1}{2(3 + 2x_n)} \langle 1$]

So by induction given sequence is bounded above by $\frac{3}{2}$. It is bounded below by 1. It is bounded.

Hence by monotone convergence theorem.

Given sequence is Convergent.

Let the limit of sequence is *l*.

$$\text{So } \lim x_{n+1} = \lim \frac{4 + 3x_n}{3 + 2x_n}$$

$$\lim x_{n+1} = \frac{4 + 3\lim x_n}{3 + 2\lim x_n}$$

$$\ell = \frac{4 + 3\ell}{3 + 2\ell}$$

$$\Rightarrow \ell = \pm\sqrt{2}$$

Since $\ell \le x_n \ \forall n \in N$,

So, $\ell = \sqrt{2}$

1.11. Nested Interval Theorem (Cantor's Intersection Theorem)

If we have a sequence $<I_n = [x_n, y_n]>$ of closed interval s.t. $I_{n+1} \subset I_n \ \forall n \in N$, and $\lim [y_n - x_n] = 0$ then $\bigcap_{n=1}^{\infty} I_n$ Consists of exactly one point.

Proof :

We have $I_{n+1} = [x_{n+1}, y_{n+1}] \subset I_n = [x_n, y_n] \quad \forall\, n \in N$

$\Rightarrow\; x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \quad \forall\, n \in N$

We conclude that sequence $< y_n >$ is monotonically decreasing bounded below by x_1 so it is converges to its greatest lower bound. Similarly $< x_n >$ is monotonically increasing bounded above by y_1

So it is converges to its least upper bound.

But it is given that

$\lim (y_n - x_n) = 0$

so $\lim y_n = \lim x_m = \xi$ (say)

So we have

$x_n \leq \xi \leq y_n \;\forall\, n$

$\Rightarrow \xi \in [x_n, y_n]$

$\Rightarrow \xi \in I_n \;\forall\, n$

$$\Rightarrow \xi \in \bigcap_{i=1}^{\infty} I_n$$

Let $\eta_1 \in \bigcap_{n=1}^{\infty} I_n$

and $\xi = \eta$

$0 \leq |\xi - \eta| \leq |y_n - x_n| \quad \forall\, n \in N$

Since $\lim |y_n - x_n| = 0$

So $|\xi - \eta| = 0$

or $\xi = \eta$

Thus $\bigcap_{n=1}^{\infty} I$ Contains exactly one point.

Monotone Subsequence Theorem :

Definition : Let $< x_n >$ is any sequence then x_m is called the peak in $< x_n >$ if $x_m \geq x_n \quad \forall\, n \geq m$.

Theorem1 : Every sequence has a monotonic subsequence

Proof :

We proof this theorem with two cases.

Let < x_n> be any sequence.

Case (a) : When < x_n > has finite no (possibly zero) of peaks. Let these are X_{n_1}, X_{n_2}, ... X_{n_r} with $n_1 < n_2 < .... < n_r$. Now there are no other peaks in < x_n > Take the term $X_{n_{r+1}}$ with n_{r+1} immediate comes after n_r. It is not peak so we have

$X_{n_{r+2}}$ s.t. $X_{n_{r+3}} > X_{n_{r+2}}$

Now $X_{n_{r+2}}$ is not a peak so we have $X_{n_{r+3}} \rangle X_{n_{r+2}}$

Continue this manner we have a monotonic subsequence $X_{n_{r+1}} \langle X_{n_{r+2}} \langle$ .....

Case : When < x_n> has infinitely no. of peaks.

Let, X_{n_1}, X_{n_2}, ... X_{n_r} ..... are infinitely many peaks with

$n_1 < n_2 < .... < n_r < ......$

Then by definition of peak we have

$X_{n_1} \geq X_{n_2} \geq ..... \geq X_{n_r} \geq .....$

So the subsequence $X_{n_1} \geq X_{n_2} \geq ..... \geq X_{n_r} \geq .....$ is monotonic.

Theorem 2 : Every bounded sequence has a convergent subsequence.

Proof :

Let < x_n > be any bounded sequence. By Monotone subsequence theorem < X_n> has monotone subsequence $\langle X_{n_k} \rangle$ (say)

Since <x_n> is bounded so its subsequence $\langle X_{n_k} \rangle$ is also bounded. Now $\langle X_{n_k} \rangle$ is a subsequence of <x_n> s.t. it is bounded and monotonic.

So by monotone convergence theorem $\langle X_{n_k} \rangle$ is convergent.

Subsequence $\langle X_{n_k} \rangle$ of <x_n> is convergent. Hence Every convergent sequence has a convergent subsequence.

1.12. Limit point of a sequence :

Let < x_n > be any sequence. A no. a $\in$ R is a limit point of <x_n>

if all neighborhood of 'a' contains infinite no. of terms of $< x_n >$.

Or we can say that $a \in R$ is a limit point of $< x_n >$ if for Every $\in > 0$, $x_n \in]\, a \in\in, a + \in [$ for infinitely many value of n.

Note :

1. It a is a limit point of $<x_n>$ then it is not necessary that a is any term of $<x_n>$.
2. Limit point of $<x_n>$ is different from the limit of sequence.
3. $a \in R$ is a limit point of sequence $<x_n>$ if $\exists$ a subsequence $\langle x_{n_k} \rangle$ which converge to a.
4. It $<x_n>$ is a sequence and $\lim x_n = a$ then a is only limit point of $<x_n>$.
5. It $<x_n>$ is bounded sequence then set of limit point of $<x_n>$ is bounded.

IMPORTANT POINT

1. The sequence $\left\langle x_n = \frac{1}{n} \right\rangle$ has only onl limit point '0'.
2. Sequence $<(-1)^n>$ has two limit point 1 and –1
3. Sequence $< n >$ has no limit point.

1.13. Bolzano-Weirs tress Theorem for sequences :

Every bounded sequence has at least one limit point.

Proof :

We take the range set $x = \{x_n \mid n \in N\}$ of bounded sequence $< x_n >$. Then X is bounded.

If X is finite. Then $\exists\, a \in R$ st. for infinitely many value of n, $x_n = a$

So for given $\in > 0$, $x_n \in]\, a - \in, a + \in [$ for infinitely many value of n. Thus Every neighborhood of a contains infinitely many terms of $<x_n>$. Hence a is limit point of $<x_n>$.

If X is infinite. Now X is infinite bounded set. Therefore X has one limit point 'a' (say) (By Bolzano weirs tress theorem). So Every neighborhood of 'a' contains infinitely many Clements of X. So we can say that given $\in > 0$ $x_n \in (a - \in, a + \in)$ for infinitely many value of n. Thus a is the limit point of $<x_n>$

Hence Every bounded sequence has at least one limit point.

Cauchy Sequence :

A sequence $<x_n>$ is said to be Cauchy sequence if to each $\in >0$ $\exists$ a + ve integer m $(\in)$ s.t.

$$|x_n - x_m| < \in \quad \forall \; n \geq m$$

or $|x_r - x_s| < \in \quad \forall\ r, s \geq m$

or $|x_{n+p} - x_n| < \in \quad \forall\ n \geq m$ and $\forall\ p \geq 1$

Some Cauchy sequence are

(i) $\left\langle \frac{1}{n} \right\rangle$ is a Cauchy sequence.

(ii) $\left\langle \frac{1}{n^2} \right\rangle$ is a Cauchy sequence.

(iii) $\left\langle \frac{(-1)^n}{n} \right\rangle$ is a Cauchy sequence.

(iv) $< n^2 >$ is not a Cauchy sequence.

(v) $<(-1)^n>$ is not a Cauchy sequence.

Theorem 1 : Every Cauchy sequence is bounded.

Proof :

It $<x_n>$ is a Cauchy sequence and $\in = 1$ by definition we know to each $\in > 0$ $\exists$a + ve integer m s.t. $|x_n - x_m| < \in \quad \forall\ n \geq m$.

So, $|x_n - x_m| < 1 \quad \forall\ n \geq m$.

$\Rightarrow \quad x_{m-1} < x_n < x_{m+1} \quad \forall\ n \geq m$.

Take $\quad r = \min\{x_1, x_2, \ldots.., x_{m-1}, x_m - 1\}$

$\quad s = \max\{x_1, x_2, \ldots.., x_{m-1}, x_m + 1\}$

So, $r \leq x_n \leq s \ \forall\ n$

Thus Every Cauchy sequence $< x_n >$ is bounded. Converse of the above theorem is not true. take the sequence $<(-1)^n>$. This sequence is bounded but not Cauchy sequence.

1.14. Cauchy General Principle of Convergence :

Every sequence is convergent iff it is Cauchy sequence

Proof :

Let $<x_n>$ is a sequence which is converge to a.

So, for given $\in > 0$, $\exists$ a + ve integer m s.t.

$$|x_n - a| \langle \frac{\in}{2} \quad \forall\ n \geq m$$

If we take n = m then

$$|x_m - a| < \frac{\in}{2}$$

Now, $|x_n - x_m| = |x_n - x_m + a - a|$

$= |x_n - a - (x_m - a)|$

$\leq |x_n - a| + |x_m - a|$

$\langle \frac{\in}{2} + \frac{\in}{2}$ from above

$= \in$

Thus we have $|x_n - x_m| \in \ \forall \ n \geq m$

Hence by definition $<x_n>$ is a Cauchy sequence.

Conversely :

Let $< x_n>$ is a Cauchy sequence to each $\in > 0$ $\exists$ a + ve integer m ($\in$) s.t.

$$|x_n - x_m| \langle \frac{\in}{2} \quad \forall\ n \geq m \qquad \text{---(1)}$$

Now, $< x_n >$ is Cauchy sequence

$\Rightarrow <x_n>$ is bounded

$\Rightarrow <x_n>$ has at least one limit point a (say)

(Bolzano-weirs tress theorem)

$\because$ a is limit point of $<x_n> \Rightarrow x_n \in \left(a - \frac{\in}{3}, a + \frac{\in}{3}\right)$ for infinitely many value of n (By definition of limit point)

There exist r > m s.t. $x_r \in \left(a - \frac{\in}{3}, a + \frac{\in}{3}\right)$

$$\Rightarrow a - \frac{\in}{3} \langle x_r \langle a + \frac{\in}{3}$$

$$\Rightarrow |x_r - a| \langle \frac{\in}{3} \qquad \text{---(2)}$$

$\because$ r > m s from (1)

$$|x_r - x_m| \langle \frac{\in}{3} \qquad \text{---(3)}$$

Now, $|x_n - a| = |x_n - a + x_m - x_m + x_r - x_r|$
$\leq |x_n - x_m| + |x_r - x_m| + |x_r - a|$

$\langle \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3}$

$= \in \quad \forall \; n \geq m$

Thus we have $|x_n - a| < \in \quad \forall \; n \geq m$

Hence $<x_n>$ Converges to a.

EXAMPLE

1. Prove that the sequence $\left\langle x_n = \frac{1}{n} \right\rangle$ is a Cauchy sequence.

Sol. :

Consider $\in > 0$ is given.

$$|x_n - x_m| = \left| \frac{1}{n} - \frac{1}{m} \right|$$

$$= \left| \frac{1}{m} - \frac{1}{n} \right| \quad \text{For} \quad n \geq m$$

$$= \frac{1}{m} - \frac{1}{n}$$

$$\langle \frac{1}{m} \langle \in \text{ if } \frac{1}{\in} \langle m$$

So, choose $m \rangle \frac{1}{\in}$

Thus $\exists$ + ve integer m s.t.

$|x_n - x_m| < \in \quad \forall \; n \geq m$

Hence $\left\langle x_n = \frac{1}{n} \right\rangle$ is a Cauchy sequence.

2. Prove that the sequence $\left\langle x_n = \frac{n+1}{n} \right\rangle$ is a Cauchy sequence?

Sol. :

Suppose $\in > 0$ is given

$$|x_n - x_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|1 + \frac{1}{n} - 1 - \frac{1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right|$$

$$= \left|\frac{1}{m} - \frac{1}{n}\right| \quad , \quad \text{For } n \geq m$$

$$\langle \frac{1}{m} \langle \in \quad \text{if} \quad \frac{1}{\in} \langle m$$

So, we choose $m \langle \frac{1}{\in}$

Thus $\exists$ a + ve integer m s.t. $|x_n - x_m| < \in \quad \forall \; n \geq m$

Hence $\left\langle x_n = \frac{n+1}{n} \right\rangle$ is a Cauchy sequence .

Others Important Theorems :

Theorem 1 : Cauchy first theorem on limits :

Let $<x_n>$ be a sequence s.t. $\lim x_n = a$

Then $\frac{x_1 + x_2 + \dots\dots + x_n}{n} = a$

Proof :

First suppose $y_n = x_n - a$

taking limit as $n \to \infty$

$\lim y_n = \lim x_n - a$

$= a - a$

$= 0$

Now, $y_1 = x_1 - a$

$y_2 = x_2 - a$

$y_n = x_n - a$

So, $x_1 + x_2 + \text{-----}\; x_n = (y_1 + a) + (y_2 + a) + \text{----} + (y_n + a)$

$\Rightarrow \quad x_1 + x_2 + \text{-----}\; x_n = (y_1 + y_2 + \text{-----} + y_m) + n\,a$

$$\Rightarrow \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \frac{y_1 + y_2 + \ldots\ldots + y_n}{n} + a$$

To prove the theorem it is sufficient to prove

$$\lim \frac{y_1 + y_2 + \ldots\ldots + y_n}{n} = 0 \qquad \text{---(1)}$$

Sequence $< y_n >$ converges to 0 and Every convergent sequence is bounded so. $< y_n>$ is bounded. So $\exists$ a natural no. N s.t.

$|y_n| < N \quad \forall$ n. ---(2)

For, (1) , $\left|\frac{y_1 + y_2 + \ldots\ldots + y_n}{n} - 0\right| = \frac{1}{n}|y_1 + y_2 + \ldots\ldots + y_n|$

$$= \frac{1}{n}|y_1 + y_2 + \ldots\ldots + y_m + y_{m+1} + \ldots\ldots + y_n|$$

$$\leq \frac{1}{n}\{|y_1| + |y_2| + \ldots\ldots + |y_m| + |y_{m+1}| + \ldots\ldots + |y_n|\}$$

Since $\lim y_n = 0$, so to each $\in > 0$, $\exists$ a + ve integer m s.t.

$$|y_n| < \frac{\in}{2} \quad \forall n \geq m \qquad \text{---(3)}$$

So by using (2) and (3) we have

$$\left|\frac{y_1 + y_2 + \ldots\ldots + y_n}{n}\right| < \frac{Nm}{n} + \frac{(n - m)\in}{2n} \quad \forall n \geq m$$

$$= \frac{Nm}{n} + \left(1 - \frac{m}{n}\right)\frac{\in}{2}$$

$$< \frac{Nm}{n} + \frac{\in}{2} \qquad \left[\because 1 - \frac{m}{n} < 1,\ n \geq m\right]$$

$$\left|\frac{y_1 + y_2 + \ldots\ldots + y_n}{n}\right| < \frac{Nm}{n} + \frac{\in}{2}$$

Take a + ve integer $M < \frac{2mN}{\in}$ so that $\frac{mN}{\in} < \frac{\in}{2}$ Where $n > M$

$M_1 = \max\{m, M\}$

$$\left|\frac{y_1+y_2+........+y_n}{n}\right| \langle \frac{\in}{2} + \frac{\in}{2} = \in \text{, for } n > M_1$$

So $\lim \dfrac{y_1+y_2+........+y_n}{n} = 0$

Hence $\lim \dfrac{x_1+x_2+........+x_n}{n} = a$

Converse of the Cauchy's first theorem on limits is not true. we consider $<x_n = (-1)^n>$

$$\frac{x_1+x_2+.......+x_n}{n} = \begin{cases} 0 \text{ if n is even} \\ -\frac{1}{n} \text{ if n is odd} \end{cases}$$

$\therefore \lim \dfrac{x_1+x_2+........+x_n}{n} = 0$

But $<x_n>$ is not convergent.

Theorem 2: Cauchy's Second theorem on limit :

Let $<x_n>$ be a sequence s.t. $x_n > 0 \ \forall$ n and $\lim x_n = a$ then $\lim (x_1+x_2+........+x_n)^{\frac{1}{n}} = a$

Proof :

Consider $<y_n = \log n>$

$\lim y_n = \lim \log x_n$

$= \log a$ ---(1) $[\because \lim x_n = a]$

$\because \lim \left[\dfrac{y_1+y_2+.......+y_n}{n}\right] = \lim y_n$ [Cauchy first theorem on limits]

$\therefore \lim \left(\dfrac{y_1+y_2+.......+y_n}{n}\right) = \log a$ from (1)

$\lim \left(\dfrac{\log x_1+\log x_2+.......+\log x_n}{n}\right) = \log a$ [$(y_n = \log x_n$ $\forall n \in m$

$$\lim \frac{\log(x_1 \,.\, x_2 \,.....\, x_n)}{n} = \log a$$

$$\lim \log(x_1 \,.\, x_2 \,.....\, x_n)^{\frac{1}{n}} = \log a$$

$$\Rightarrow \lim (x_1 \,.\, x_2 \,.....\, x_n)^{\frac{1}{n}} = \log a$$

Theorem 3 : If $<x_n>$ is a sequence of positive terms and $\lim \frac{x_{n+1}}{x_n} = a$ then

$$\lim (x_n)^{\frac{1}{n}} = a$$

Proof :

We consider a sequence

$$y_1 = x_1 \,,\, y_n = \frac{x_n}{x_{n-1}} \quad \forall\, n \geq 2 \qquad \text{---(1)}$$

Then we have

$$y_1, y_2 ....y_n = x_1 \frac{x_2}{x_1} \cdot \frac{x_3}{x_2} \,......\frac{x_{n-1}}{x_{n-2}} \cdot \frac{xn}{x_{n-1}}$$

$$y_1, y_2 \,...........\, y_n = x_n \qquad \text{---(2)}$$

Now, $\lim \frac{x_{n+1}}{x_n} = a$

$$\Rightarrow \lim \frac{x_n}{x_{n-1}} = \lim y_n = a$$

Sequence (1) is + ve term sequence as $x_n > 0 \;\forall$ n thus by Cauchy second theorem on limits we have

$$\lim(y_1 \cdot y_2 \,.......y_n)^{\frac{1}{n}} = a$$

$$\lim(x_n)^{\frac{1}{n}} = a \qquad \text{from (2)}$$

Theorem 4 : Cesaro's theorem

If sequence $<x_n>$ converges to a_1 and sequence $<y_n>$ converges to a_2 then

$$\lim \frac{x_1 y_n + x_2 y_{n-1} + \ldots\ldots + x_n y_1}{n} = a_1 a_2$$

Proof :

Given $\lim x_n = a_1$

and $\lim y_n = a_2$

Consider $x_n = a_1 + Z_n$ ---(1)

and $|Z_n| = t_n$ ---(2)

$\lim x_n = a_1 + \lim Z_n$

$a_1 = a_1 + \lim Z_n$

$\lim Z_n = 0$

and $\lim t_n = 0$

Since $\lim t_n = 0$, So $\lim \dfrac{t_1 + t_2 + \ldots\ldots + t_n}{n} = 0$ (By Cauchy first theorem) ---(3)

Now, $\dfrac{1}{n}\left[x_1 y_n + x_2 y_{n-1} + \ldots\ldots + x_n y_1\right]$

$$= \frac{1}{n}\left[(a_1 + Z_1) y_n + (a_1 + Z_2) y_{n-1} + \ldots\ldots + (a_1 + Z_n) y_1\right] \quad \text{from (1)}$$

$$= \frac{1}{n}\left[a_1 y_n + Z_1 y_n + a_1 y_n + Z_2 y_{n-1} + \ldots\ldots + a_1 y_1 + Z_n y_1\right]$$

$$= \frac{1}{n}\left[a_1 (y_1 + y_2 + \ldots\ldots + y_n) + (Z_1 y_n + Z_2 y_{n-1} + \ldots\ldots + Z_n y_1)\right] \text{---(4)}$$

$$\left|\frac{Z_1 y_n + Z_2 y_{n-1} + \ldots\ldots + Z_n y_1}{n}\right| \le \frac{|Z_1||y_n| + |Z_2||y_{n-1}| + \ldots\ldots + |Z_n||y_1|}{n} \quad \text{---(5)}$$

Since $<y_n>$ Converges a_2 so it is bounded. $\exists$ a $M \in R^+$ s.t.

$|y_n| < M \ \forall n$ ---(6)

So from (5) & (6)

$$0 \le \frac{1}{n}\left|Z_1 y_n + Z_2 y_{n-1} + \ldots\ldots + Z_n y_1\right| \ \langle\ \frac{M}{n}\left\{|Z_1| + |Z_2| + \ldots\ldots + |Z_n|\right\}$$

$$= \frac{M}{n}\left\{t_1 + t_2 + \ldots\ldots + t_n\right\}$$

$\to 0$ as $n \to \infty$ from (3)

$$\lim_{n\to\infty} \frac{Z_1 y_n + Z_2 y_{n-1} + \ldots\ldots + Z_n y_1}{n} = 0 \qquad \text{---(7)}$$

taking lim as $n \to \infty$ of (4) we have

$$\lim \frac{[x_1 y_n + x_2 y_{n-1} + \ldots\ldots + x_n y_1]}{n} = \lim \frac{a}{n}[y_1 + y_2 + \ldots\ldots + y_n]$$

$$+ \lim \frac{Z_1 y_n + Z_2 y_{n-1} + \ldots\ldots + Z_n y_1}{n} = a_1 \lim \left[\frac{y_1 + y_2 + \ldots\ldots + y_n}{n}\right]$$

$$+ \lim \frac{Z_1 y_n + Z_2 y_{n-1} + \ldots\ldots + Z_n y_1}{n}$$

$$\lim \left[\frac{x_1 y_n + x_2 y_{n-1} + \ldots\ldots + x_n y_1}{n}\right] = a_1 a_2 + 0$$

$$[\because (\lim y_n = a_2 \text{ so, } \lim \frac{y_1 + y_2 + \ldots\ldots + y_n}{n} = a_2$$

and from (7)

Here,

$$\lim \frac{x_1 y_n + x_2 y_{n-1} + \ldots\ldots + x_n y_1}{n} = a_1 a_2$$

EXAMPLE

1. Prove that $\lim \dfrac{1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{3}}{n} = 0$

Sol. :

From Cauchy first theorem on limit it $\lim x_n = a$ then

$$\lim \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = a$$

$$\lim x_n = \lim \frac{1}{n} = 0$$

So, $\lim \dfrac{1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}}{n} = 0$

2. **Show that** $\lim\left[\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+.........+\dfrac{1}{\sqrt{n^2+n}}\right]=1$

Sol. :

From sandwich theorem if $<x_n>$, $<y_n>$, $<Z_n>$ are sequences s.t. $x_n \le y_n \le Z_n \quad \forall$ n and $\lim x_n = \lim Z_n = a$ then $\lim y_n = a$

take $y_n = \dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+.........+\dfrac{1}{\sqrt{n^2+n}}$

$\Rightarrow$ take $\left\langle x_n = \dfrac{n}{\sqrt{n^2+n}}\right\rangle, \left\langle z_n = \dfrac{n}{\sqrt{n^2+1}}\right\rangle$ two

Sequence then

$$y_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + ...+ \frac{1}{\sqrt{n^2+n}}$$

$$\le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + ...+ \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}$$

$$\Rightarrow y_n \le \frac{n}{\sqrt{n^2+1}} \qquad ---(1)$$

and $\dfrac{1}{\sqrt{n^2+n}} + \dfrac{1}{\sqrt{n^2+n}} + ...+ \dfrac{1}{\sqrt{n^2+n}}$

$$\le \frac{1}{\sqrt{n^2+1}} \; \frac{1}{\sqrt{n^2+2}} + ...+ \frac{1}{\sqrt{n^2+n}} = y_n$$

$$\frac{n}{\sqrt{n^2+n}} \le y_n \qquad ---(2)$$

From (1) & (2)

$x_n \le y_n \le Z_n$

and $\lim x_n = \frac{n}{\sqrt{n^2+n}} = \lim \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$

$\lim Z_n = \lim \frac{n}{\sqrt{n^2+1}} = \lim \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$

$\lim x_n = \lim Z_n = 1$

By Sandwich thearem

$$\lim y_n = \lim \left[\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + ...+ \frac{1}{\sqrt{n^2+n}} \right] = 1$$

3. Prove that $\lim\limits_{n\to\infty} \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} ....\frac{n}{n-1} \right)^{\frac{1}{n}} = 1$

Sol. :

Take $x_n = \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} ....\frac{n}{n-1} \right)$

Then $x_{n+1} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} ....\frac{n}{n+1} \cdot \frac{n+1}{n}$

$$\lim \frac{x_{n+1}}{x_n} = \lim \frac{\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} ....\frac{n}{n-1} \cdot \frac{n+1}{n}}{\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} ....\frac{n}{n-1}}$$

$$\lim \frac{n+1}{n} = \lim \left(1+\frac{1}{n}\right) = 1$$

Since we know that if $<x_n>$ is a sequence of positive terms and $\lim \frac{x_{n+1}}{x_n} = a$ then

$\lim (x_n)^{\frac{1}{n}} = a$

So $\lim x_n = 1$ i.e. $\lim \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots\cdots \frac{n}{n-1}\right) = 1$

4. Find the limit of sequence $\left\langle (n)^{\frac{1}{n}} \right\rangle$?

Sol. :

Consider $x_n = n$

$x_{n+1} = n + 1$

There fore $\lim \frac{x_{n+1}}{x_n} = \lim \left(1 + \frac{1}{n}\right) = 1$

Hence $\lim (x_n)^{\frac{1}{n}} = \lim (n)^{\frac{1}{n}} = \lim \frac{x_{n+1}}{x_n} = 1$

5. Show that $\lim \sqrt{\frac{n+1}{n}} = 1$.

Sol. :

Given $y_n = \sqrt{\frac{n+1}{n}}$

$y_n = \sqrt{\frac{n+1}{n}}$

$\Rightarrow y_n = \sqrt{1 + \frac{1}{n}}$

$\Rightarrow y_n = \sqrt{1 + \frac{1}{n}} > 1$

$\Rightarrow 1 < y_n = \sqrt{1 + \frac{1}{n}}$ ---(1)

Also $y_n = \sqrt{1 + \frac{1}{n}} < \left(1 + \frac{1}{2n}\right)$ ---(2)

From (1) & (2)

$$1 < y_n = \sqrt{\frac{n+1}{n}} \langle \left(1 + \frac{1}{2n}\right)$$

take $x_n = 1$ and $Z_n = \left(1 + \frac{1}{2n}\right)$

So, $\lim x_n = \lim 1 = 1$

and $\lim Z_n = \lim \left(1 + \frac{1}{2n}\right) = 1$

By Sandwich theorem

$\lim y_n = 1$

6. Determine $\lim \left(1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + \dots\dots + n^{\frac{1}{n}}\right)$**?**

Sol. :

take $x_n = n^{\frac{1}{n}}$

$$\lim x_n = \lim (n)^{\frac{1}{n}} = 1$$

Therefore $\lim \left[\frac{1 + (2)^{\frac{1}{2}} + (3)^{\frac{1}{3}} + \dots\dots + (n)^{\frac{1}{n}}}{n}\right]$

$= \lim x_n = 1$ (By Cauchy first theorem on limit.)

EXERCISE : 1

1. Write the n^{th} term of the following sequence --

(a) $< 2, 4, 6, \dots\dots >$ (b) $\left\langle \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\dots \right\rangle$

(c) $<1, 1, 1, \dots\dots>$ (d) $<1^3, 2^3, 3^3, \dots\dots>$

2. By definition, show that the sequence $\left\langle \frac{1}{n} \right\rangle^2$ Converges to '0'?

3. By definition, show that the sequence $\left\langle \frac{1}{x^2} \right\rangle$ converges to '0'.

   (a) Sequence $\left\langle \frac{3n+4}{2n+1} \right\rangle$ Converges to $\frac{3}{2}$

   (b) Sequence $\left\langle \frac{1}{n^2+1} \right\rangle$ Converges to 0

   (c) Sequence $\left\langle \frac{n^2-1}{2n^2+3} \right\rangle$ Converges to $\frac{1}{2}$

   (d) Sequence $\left\langle \frac{2n}{n+3} \right\rangle$ Converges to 2.

4. By definition prove that

   (a) Sequence $<-n^2>$ diverges to $-\infty$

   (b) Sequence $<2^n>$ diverges to ∞

5. Find the limit of the following sequences

   (a) $\lim \frac{\sin n}{n}$ (b) $\lim \frac{n^2+3n+5}{2n^2+5n+7}$

   (c) $\lim \frac{1}{3^n}$

6. Show that the sequence $\left\langle \frac{2n^2+3}{n^2+1} \right\rangle$ Convergent ?

7. Show that the sequence $\left\langle \frac{(-1)^n n}{n+1} \right\rangle$ divergent ?

8. Show that the sequence $\left\langle \frac{2n}{n+4(n)^{\frac{1}{2}}} \right\rangle$ is convergent.

9. Show that the sequence $\left\langle \sqrt{n+1} - \sqrt{n} \right\rangle$ is convergent.

10. Prove that the sequence < r^n > is converges to 0 when | r | < 1.

11. Show that the sequence $x_1 = \sqrt{2}$, $x_{x+1} = \sqrt{2 + x_n}$ Converges to + ve roots of the $x^2 - x - 2 = 0$.

12. Show that < x_n> defined be $x_n = \frac{1}{\angle 1} + \frac{1}{\angle 2} + \ldots\ldots.. + \frac{1}{\angle n}$ is Convergent.

13. Show that the sequence <x_n> defined by $x_n = \frac{1}{\angle 1} + \frac{1}{\angle 2} + \ldots\ldots.. + \frac{1}{\angle n}$ is Convergent.

14. Show the sequence $x_1 = a > 0$ $x_{n+1} = \sqrt{\frac{ab^2 + x_n^2}{a+1}}$, $b > a$,

    $n \geq 1$ Converges to b.

15. By definition show that the following sequences are Cauchy sequence :

    (a) $\left\langle \frac{(-1)^n}{n} \right\rangle$ (b) $\left\langle 1 + \frac{1}{\angle 2} + \ldots\ldots. + \frac{1}{\angle n} \right\rangle$

16. Prove that the sequence <x_n> where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}$ can not converges.

17. Let a sequence of positive numbers <x_n> defined by $x_n = \frac{1}{2}\left(x_{n-1} + x_{n-2}\right) \forall n \geq 3$ then

    prove that the sequence is converges and has the limit $\frac{1}{3}\left(x_1 + 2x_2\right)$.

18. Find the limit point of the sequence $\left\langle \left(1 + \frac{1}{n}\right)^{n+1} \right\rangle$

19. (a) Show that

$$\lim\left[\frac{1}{\sqrt{2n^2+1}} + \frac{1}{\sqrt{2n^2+2}} + \text{.......}+ \frac{1}{\sqrt{2n^2+n}}\right] = \frac{1}{2}$$

(b) Show that

$$\lim\left[\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \text{.......}+ \frac{1}{\sqrt{n+n}}\right] = \infty$$

20. (a) Prove that $\lim \frac{x^n}{\angle n} = 0, \quad x \in R$

(b) Prove that $\lim \frac{\angle n}{n^n} = 0$

21. Show that $\lim\left[\frac{\angle 3n}{(\angle n)^3}\right]^{\frac{1}{n}}$

ANSWERS : 2

(1) (a) 2n (b) $\frac{1}{2n}$

(c) 1 (d) n^3

5. (a) 0 (b) $\frac{1}{2}$

(c) 0

18. e.

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Chapter 2

REAL NUMBER SYSTEM

2.1. Introduction :

In this chapter we shall study some important properties of real numbers systems. Firstly, we discuss the algebraic property of real number system. We discuss the absolute value notion which is depend on the order property of R. We will also study Nested interval property and we will also use this property for proving the uncountability of R.

In short, we shall study the basic properties of real number systems is three categories.

(a) Field axiom

(b) Order axiom

(c) Completeness axiom.

(a) Field axiom : Let R is set of real number with two binary operation addition '+' and 'multiplication'. satisfy the following axiom.

(i) $\forall$ a, b $\in$ R, a + b $\in$ R i.e. R is closed w.r.t. addition.

(ii) $\forall$ a, b, c $\in$ R, a + (b + c) = (a + b) + c, addition is associative.

(iii) $\forall$ a $\in$ R, $\exists$ an element '0' in R called zero clement s.t. a + 0 = 0 + a = a.

(iv) For each a in R $\exists$ an element – a in R s.t. a + (– a) = 0 = – a + a

(v) $\forall$ a, b $\in$ R, a . b $\in$ R i.e. R is closed w.r.t. multiplication.

(vi) $\forall$ a, b $\in$ R, a . b = b . a , R is commutative w.r.t. multiplication.

(vii) $\forall$ a, b, c $\in$ R, a . (b . c) = (a . b) . c i.e. multiplication is associative.

(viii)(Existence of unit element) $\forall$ a in R, $\exists$ an element 1 $\neq$ 0 in R s.t. a . 1 = 1 . a = a

(ix) (Existence of inverse) Each non-zero element a in R possess multiplicative inverse i.e.

$\forall$ a, $\in$ R, a $\neq$ 0 $\exists$ an element b $\in$ R s.t. ab = 1, b is called multiplicative. inverse of a and is denoted by a^{-1}

(x) $\forall$ a, b, c , a . (b + c) = a . b + a . c and (b + c) . a = b . a + c . a i.e.

multiplication is distributive over addition. Now we can say that (R, +, .) is a field. The above axioms are called field axiom. Infield axioms first four axiom are related to addition and the axioms from (v) to (ix) are related to multiplication. In (iii) axiom element '0' ie. Zero element is unique for all elements. In (iv) axiom '– a' is called negative of a.

Theorem :

1. If $a, b \in R$ s.t. $b + a = a$ then $b = 0$
2. If $a, b \in R$ s.t. $a, b \neq 0$ and $a \cdot b = b$

Then $a = 1$

3. $\forall\ a \in R, a \cdot 0 = 0$
4. $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

Proof :

1. $b = b + 0$ From field axiom (iii)

$b = b + [a + (-a)]$ From field axiom (iv)

$b = [(b + a)] - a$ From field axiom (ii)

$b = a + (-a)$ [Given $b + a = a$]

$b = 0$ From field axiom (iv)

2. $a = a \cdot 1$ From field axiom (viii)

$a = a \cdot \left(b \cdot \frac{1}{b}\right)$ From field axiom (ix)

$a = (a \cdot b)\left(\frac{1}{b}\right)$ From field axiom (vii)

$a = b \cdot \frac{1}{b}$ (given $ab = b$)

$a = 1$

3. $a + a\,0 = a + a\,0$

$= a \cdot 1 + a\,0$ From field axiom (viii)

$= a\,(1 + 0)$ From field axiom (x)

$= a \cdot 1$

$= a$

$\Rightarrow$ $a \cdot 0 = a$ From theorem (i)

4. To prove this theorem it is sufficient is show if $a \neq 0$ then $b = 0$

take $a \neq 0$

$a \cdot b = 0$

$a^{-1}(a \cdot b) = a^{-1} \cdot 0$

$(a^{-1} \cdot a)\, b = 0$ From field axiom (viii) and

From theorem (3)

$1 . b = 0$ From field axiom (ix)

$b = 0$

(I) Subtraction in R :

If $a . b \in R$, then the operation subtraction is denoted by $a - b$ and defined as $a - b = a + (- b)$

For subtraction between a & b, $a - b \neq b - a$

(II) Division in R :

If $a, b \in R, (b \neq 0)$ the division is denoted by $\frac{a}{b}$ or $a \div b$ or a/b and defined as $\frac{a}{b} = a \cdot \frac{1}{b}$

if $b = 0$ the division is not allowed

2.2. Important properties of real numbers :

Real numbers have some important properties which are necessary for us

(i) If $a + b = a + c$ then $b = c$

(ii) $-(- a) = a$

(iii) If $a \neq 0$ and $a . b = a . c$ then $b = c$

(iv) If $a \neq 0$ then $\frac{1}{1/a} = a$

(v) If a & b are non-zero real numbers then $a . b$ is also non-zero

(vi) For $a, b \in R$, $a (-b) = -(a .b)$ and $(- a) . b = -(a . b)$

(vii) For $a, b \in R$, $(- a) (-b) = ab$

(viii) From $a, b \in R$, $-(a + b) = - a - b$

(ix) For two non-zero real number a, b, $\frac{1}{a \cdot b} = \left[\frac{1}{a}\right] \cdot \left[\frac{1}{b}\right]$

(x) If a is a non-zero real number and is any real number then $x = b/a$ in R is a unique solution of the equation $ax = b$

(xi) If $a, b \in R$ then $x = b - a \in R$ is a unique solution for the equation $x + a = b$

2.3. Integral power of real number :

Let n is any positive integer and $a \in R$ then we define, particularly $a^1 = a$, $a^2 = a . a$, $a^3 = a^2.a = a . a . a$ .... In general $a^n = a . a . a......$ n times.

We write $a^0 = 1$

If $a \neq 0$ then $a^{-n} = (a^n)^{-1} = (a^{-1})^n$.

(b) Order axioms for real numbers :

The following axioms are satisfied by order relation greater than (>) between two real numbers :

(i) Let $a, b \in R$, then only one of the following.

$a = b$, $a > b$, $a < b$ (Trichotomy)

(ii) If $a, b, c \in R$ then $a > b$, $b > C \Rightarrow a > c$ (transitivity)

(iii) If $a, b, c \in R$ then $a > b \Rightarrow a + c = b + c$ (monotone property for addition)

(iv) Let $a, b, c \in R$. If $a > b$, $c > 0$ then $ac > bc$. (Monotone property for multiplication)

Now from above we can say that field of real $(R, +, .)$ is an ordered field $(R, +, ., >)$

The system C of all complex number is a field but not an ordered field.

2.4. Some Important Definition :

(i) $a \in R$ is + ve if $a > 0$

(ii) $a \in R$ is – ve if $a < 0$.

We denote the set of all + ve real numbers by R^+ and set of all – ve real numbers by R^-

So, $R = R^+ \cup R^- \cup \{0\}$.

(iii) Let a and b are any two real number then $a \leq b$ if $a < b$ or $a = b$

(iv) If $a, b \in R$ then $a \geq b$ if $a > b$ or $a = b$.

(v) Between two real numbers a and b the order relation 'less then' (<) is defined as $a < b$ if $b > a$

2.5. Some Properties for order relation :

(i) If a is any + ve real number and b is any negative real number then $a > b$.

(ii) $\forall\, a \in R$, only one of the following is true ---

$a < 0$, $a = 0$, $-a < 0$

(iii) $\forall\, a \in R$, only one of the following is true ---

$a > 0$, $a = 0$, $-a > 0$

(iv) If $a, b \in R^+$ then $a + b > 0$ and $ab > 0$

(v) If a , b are any two negative real number then $a + b$ and ab are negative real number and + ve real number respectively.

(vi) a is less than b and b is less than c then a is less than c. i.e. if $a < b$ and $b < c$ then $a < c$.

(vii) $a < b$ iff $a + c < b + c$

(viii) If $a < b$ and c $a \in R$, R^- then $a\,c > bc$

(ix) $a \in R^-$ iff $-a \in R^+$, $a \in R^+ \Leftrightarrow -a \in R^-$.

(x) a is greater than b $(a > b)$ iff – a is less then – b $(-a < -b)$

(xi) If a > b and b > 0 then $0 < \frac{1}{a} < \frac{1}{b}$.

(xii) $a \neq 0$ then $a^2 > 0$

(xiii) If $a, b \in R^+$ and $a > b$ then $a^2 > b^2$.

If $a, b \in R^-$ and $a < b$ then $a^2 > b^2$.

2.6. Some Subset of R :

1. Natural numbers set (N) :

Inductive Set : Any subset M of R is said to be Inductive set if (i) $1 \in M$ and (ii) $r \in M \Rightarrow r + 1 \in M$.

Natural number set N is the smallest inductive subset of R.

From above hypothesis $1 \in N \Rightarrow 1 + 1 = 2 \in N$, $2 \in N \Rightarrow 2 + 1 = 3 \in N$, $3 \in N \Rightarrow 3 + 1 = 4 \in N$........

Thus we have $N = \{1, 2, 3, ...\}$

2.7. Mathematical Induction Principle :

Any preposition P (k) is true $\forall k \in N$ provided

(i) When k = 1, the preposition is true i.e. P (1) is true

(ii) If P(n) is true $\forall n \in N$ then P(n + 1) is true.

2. Set of Integer Z :

$Z = \{0, \pm 1, \pm 2, ...\}$ C R is

said the set of integer. We have $N \subset Z \subset R$.

3. Set of Rational Numbers :

We denote the set of rational number by Q and defined as

$$Q = \left\{ \frac{p}{q} \mid p, q \in Z, q \neq 0 \right\}$$

We have $N \subset Z \subset Q \subset R$

4. Set of Irrational Numbers :

$a \in R$ is irrational number if it is not rational number. Thus R – Q set is irrational numbers set.

EXAMPLE

1. Prove that $\sqrt{2}$ is irrational number ?

Sol. :

Consider $\sqrt{2}$ is rational

So, $\sqrt{2} = \frac{p}{q}$ where, p, q $\in$ Z, q $\neq$ 0

and p, q have no factor in common.

Now, $\frac{p}{q} = \sqrt{2} \Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q$

$\Rightarrow p^2$ is even

p should be even.

taken p = 2 m

Since $P^2 = 2\ q^2$

So $\quad 4\ m^2 = 2\ q^2$

or $\quad q^2 = 2\ m^2$

So q should be even taken q = 2m

Since there is a common factor between pand q which is contradiction.

Thus $\sqrt{2}$ is irrational number.

2. Prove that $\sqrt{8}$ is not rational number ?

Sol.:

Consider $\sqrt{8}$ is rational number

So $\sqrt{8} = \frac{p}{q}$, p, q are integers prime to each other and q □≠ 0.

Since $\sqrt{8} = \frac{p}{q}$

$\Rightarrow 2 < \sqrt{8} = \frac{p}{q} < 3$

$\Rightarrow 2\ q < p < 3\ q$

$\Rightarrow 0 < p - 2\ q < q$

p – 2 q is + ve integer and less than q

So, $\sqrt{8}\,(p\text{-}2\,q) = \frac{p}{q}(p\text{-}2\,q)$

$$= \frac{p^2}{q} - \frac{2pq}{q}$$

$$= \frac{p^2}{q^2} \cdot q - 2p$$

$$= \left(\sqrt{8}\right)^2 q - 2p \qquad \left[\because \frac{p}{q} = \sqrt{8}\right]$$

= integer

Which is contradiction. Therefore $\sqrt{8}$ is not a rational number.

2.8. Intervals :

1. Closed interval : Let a, b $\in$ R s.t. a < b then we define the closed interval the set $\{x \mid a \le x \le b\}$

We denote it by [a, b]. a & b real no also lie in this set.

2. Open intervals : Let a, b $\in$ R s.t. a < b then the set $\{x \mid a < x < b\}$ is called the open interval and w denote it] a, b [or (a, b). a, b real no. do not lie in this set.

3. Semi open or Semi closed interval : The set defined by $\{x \mid a < x \le b\}$ and $\{x \mid a \le x < b\}$ are said to be semi open or semi closed and is denoted by] a, b [and [a, b[respectively.

4. Closed rays : The sets defined by $\{x \mid a \le x\}$

and $\{x \mid x \le a\}$ are called closed rays. we can write these as $[a, \infty[$ and $]-\infty, a]$ respectively.

5. Open rays : The sets defined by $\{x \mid a < x\}$ and $\{x \mid x < a\}$ are called open rays and we can write these as $]a, \infty[$ and $]-\infty, a[$ respectively.

6. Length of Intervals : The length of an interval with end point a and b (a< b) is b – a.

Thus the intervals (a, b) [a, b], (a, b], [a, b) have the length b – a. The length of intervals (a, ∞), $[a, \infty)$, $(-\infty, a)$, $(-\infty, a)]$ $(-\infty, \infty)$ is infinite. These intervals are infinite intervals.

7. Absolute Value :

The absolute value of a real no. x is denoted by |x| and defined as

$$|x| = \begin{cases} x & , \ge 0 \\ -x & , \le 0 \end{cases}$$

e.g. 5, – 5 $\in$ R so the absolute value of 5 is | 5 | = 5 since 5 > 0. for –5 the absolute value is | –5 | = – (–5) = 5 since – 5 < 0

Note : $|x| = 0 \Leftrightarrow x = 0$

Theorem on Absolute value :

It $x, y \in R$ then

(i) $|x| \geq 0$ (ii) $|x| \geq x$

(iii) $|x|^2 = x^2 = |-x|^2$ (iv) $|x| = \max\{x, -x\}$

(v) $|x| = |-x|$ (vi) $x \geq -|x|$

(vii) $|x\ y| = |x|.|y|$ (viii) $|x + y| \leq |x| + |y|$ Triangle inequality

(ix) $|x - y| \geq |\,|x| - |y|\,|$

(x) $|x - y| < \in \Rightarrow y - \in < x < y + \in$ Where $\in > 0$

Proof :

(i) Let $x \in R$, to prove $|x| \geq 0$

we know $|x| = \begin{cases} x & , \geq 0 \\ -x & , \leq 0 \end{cases}$, By definition.

If $x \geq 0$ then $|x| = x$

$\Rightarrow |x| = x \geq 0$

$\Rightarrow |x| \geq 0$

If $x \leq 0$ then $|x| = -x$

$\Rightarrow |x| = -x \geq 0$ [$\because -x$ is non -negative]

$\Rightarrow |x| \geq 0$

Hence $|x| \geq 0 \ \forall \ x \in R$

(ii) To Prove $|x| \geq x$

If $x = 0$, then we have to prove nothing

be cause $|0| = 0$ so $|x| = x$ in this case.

if $x > 0$ then $|x| = x$ by definition of absolute value.

If $x < 0$ then $|x| = -x > x$

[$\because$ x is negative so $-x$ is + ve quantity]

Hence $|x| \geq x \ \forall \ x \in R$

(iii) To Prove $|x|^2 = x^2 = |-x|^2$

If $x \geq 0$ then $|x| = x$ By definition

$\Rightarrow |x|^2 = x^2$

If $x \leq 0$ then $|x| = -x$ By definition

$\Rightarrow |x|^2 = x^2$

So, $\forall\ x \in R$, $|x|^2 = x^2$ ...(1)

If $x \geq 0$ then $-x \leq 0$ So, $|-x| = -(-x)$ By definition.

$\Rightarrow |-x|^2 = x^2$

If $x \leq 0$ then $-x \geq 0$ So, $|-x| = -x$ By definition

$\Rightarrow |-x|^2 = x^2$

So, $\forall\ x \in R, |-x|^2 = x^2$ ...(2)

So from (1) & (2)

$\forall\ x \in R, |x|^2 = x^2 = |-x|^2$

(iv) To Prove $|x| = \max\{x, -x\}$

By definition of absolute value $|x| = \begin{cases} x & , x \geq 0 \\ -x & , x \leq 0 \end{cases}$

thus in Every case $|x|$ is greater of two real no. $x, -x$.

(v) To Prove $|x| = |-x|$

From (iv) theorem we have $|x| = \max\{x, -x\}$

Now replace x by $-x$ we have $|-x| = \max\{-x, x\} = |x|$

(vi) To Prove $x \geq -|x|$

If $x \geq 0$ then by definition of absolute value we have

$|x| = x$

So, $x \geq -|x|$

If $x \leq 0$ then by definition of absolute value we have

$|x| = -x$, $-x$ is + ve quantity

$\Rightarrow -|x| = x$

Thus $\forall\ x \in R$, We have $x \geq -|x|$

(vii) To Prove $|x\,y| = |x|\,|y|$

From theorem (iii) we know $|a|^2 = a^2\ \forall\ x \in R$

So, $|x\,y|^2 = (x\,y)^2 \Rightarrow |x\,y|^2 = x^2\,y^2$

$\Rightarrow |x\,y|^2 = |x|^2\,|y|^2$

$\Rightarrow |x\,y|^2 = (|x|\,|y|)^2$

$\Rightarrow |x\,y| = \pm |x|\,|y|$

[$\because |a| \geq 0\ \forall\ a \in R$ so we take + ve sign]

(viii) To Prove $|x + y| \leq |x| + |y|$

From the theorem (iii) we know $|a|^2 = a^2\ \forall\ a \in R$

So,

$|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2\,xy \leq |x|^2 + |y|^2 + 2\,|x|\,.\,|y|$

[From theorem (ii) $|a| \geq a \ \forall a \in R$]

$= (|x| + |y|)^2$

$\Rightarrow |x + y|^2 \leq (|x| + |y|)^2$

$\Rightarrow |x + y| \leq |x| + |y|$ [$\because |a| \geq 0 \ \forall a \in R$ so we take + ve sign]

(ix) To Prove $|x - y| \geq ||x| - |y||$

$\because \quad x = x$

$\Rightarrow x = x - y + y$

$\Rightarrow |x| = |x - y + y|$

$\Rightarrow |x| \leq |x - y| + |y|$ From theorem (viii)

$\Rightarrow |x| - |y| \leq |x - y|$ ...(1)

Now, $y = y$

$\Rightarrow y = y - x + x$

$\Rightarrow |y| = |y - x + x| \Rightarrow |y| \leq |y - x| + |x|$

$\Rightarrow |y| - |x| \leq |y - x|$

$\Rightarrow -(|x| - |y|) \leq |x - y|$ ...(2)

From (1) and (2)

$|x - y| \geq \max \{|x| - |y|, -(|x| - |y|)\}$

$= ||x| - |y||$

$\Rightarrow |x - y| \geq ||x| - |y|| \quad \forall \ x . y \in R$

(x) To Prove $|x - y| < \in \Leftrightarrow y - \in < x < y + \in$

Since we know that $|x| = \max [x, -x\}$

So, $|x - y| = \max \{(x - y), (y - x)\}$ ...(1)

It is given that $|x - y| < \in$

So $|x - y| < \in \Leftrightarrow \max \{(x - y), (y - x)\} < \in$ From (1)

$\Leftrightarrow (x - y) < \in, (y - x) < \in$

$\Leftrightarrow x < y + \in, y - \in < x$

$\Leftrightarrow y - \in < x < y + \in$

Particularly if we take $y = 0$ then

we have $|x| < \in \Leftrightarrow - \in < x < \in$

EXAMPLE

1. Show that $|x - y| \leq |x| + |y|$

Sol.:

We know $\leq |x| + |y|$

So, $| x - y | = | x + (-y) |$

$\leq | x | + | -y |$ $\quad [\because | -a | = | a |]$

$= | x | + | y |$

Thus $|x - y| \leq |x| + |y|$

2. Show that $| x + y |^2 + | x - y |^2 = 2 | x |^2 + 2 | y |^2, \ \forall x, y \in R$

Sol.:

Since we have $| a |^2 = a^2 \ \forall a \in R$, so

$| x + y |^2 + | x - y |^2 = (x + y)^2 + (x - y)^2$

$= 2x^2 + 2y^2$

$= 2 | x |^2 + 2 | y |^2$

3. Let k is any positive real number and $| x - y | < k \in \ \forall \in > 0$ then show that $x = y$

Sol.:

Consider $x \neq y$

take $\in = \dfrac{1}{2k} |x - y|$

Clearly $\in > 0$ since $k \in R^+$ & $|x - y| > 0$

Now, $|x - y| < k \in \ \forall \in > 0$ (given)

$\Rightarrow | x - y | < k . \dfrac{1}{2k} | x - y |$

$\Rightarrow | x - y | < \dfrac{1}{2} | x - y |$

Which is not possible. So our assumption is not correct. Therefore we must have $x = y$.

4. Show that $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, \ y \neq 0$

Sol. :

Since we have $| a |^2 = a^2$ for absolute value of $a \in R$ So

$\left|\dfrac{x}{y}\right|^2 = \left(\dfrac{x}{y}\right)^2$

$= \dfrac{x^2}{y^2}$

$$= \frac{|x|^2}{|y|^2}$$

$$\Rightarrow \left|\frac{x}{y}\right|^2 = \left(\frac{|x|}{|y|}\right)^2$$

$$\Rightarrow \left|\frac{x}{y}\right| = \pm \frac{|x|}{|y|}$$

$$\Rightarrow \left|\frac{x}{y}\right| = \pm \frac{|x|}{|y|} \quad , y \neq 0 \text{ leaving – ve sign}$$

2.9. Bounded and unbounded sets :

Let R be a set of real numbers. Any subset x of R is said to be bounded if it is both bounded above and bounded below.

x is bounded above if $\exists$ a real no. r s.t. $r \geq x \ \forall x \in X$. If r exist then we say that r is upper bound of x. Every real numbers which are greater than r are called the upper bound of x. Thus any set which is bounded above have infinitely many upper bound. The upper bound which is minimum in all of these upper bound is called least upper bound or supremum of set x. Thus a upper bound r_1 is supremum of set x.

If any upper bound 'r' of x s.t. $r < r_1$.

x is bounded below if $\exists$ a real no. s, s.t.

$s \geq x \ \forall x \in x$. If s exist then we say that s is lower bound of x. Every real number which are less than 's' are the lower bound of x. Thus any set which is bounded below have infinitely many lower bound. The lower bound which is maximum in all of these lower bound is called greatest lower bound or infimum of set x. Thus a lower bound s_1 is infimum of set x if any lower bound s of x s.t. $s_1 < s$.

Thus we can define bounded set as

'A set x is bounded iff $\exists$ real numbers r & s, s.t. $s \leq x \leq r \ \forall x \in x$.

Now, A set x is unbounded if it is not bounded i.e. if it is not bounded above or not bounded below.

IMPORTANT POINT

(i) Empty set ϕ is bounded although it has no supremum and infimum.

(ii) Any non-empty finite subset of R is bounded.

(iii) Every singleton set is bounded. The sup. and inf. of this set is the single element of this set.

(iv) R^+ is bounded below but not bounded above.

(v) R^- is bounded above but not bounded below.

(vi) R is unbounded. It is neither bounded above nor bounded below.

(vii) I = set of integers is unbounded set .

(viii) I^+ is bounded below but not bounded above.

(ix) I^- is bounded above but not bounded below.

(x) $\left\{\frac{1}{n} \middle| n \in N\right\}$ is bounded set.

2.10. Some important properties of supremum and infimum of subset of R :

(i) Any bounded subset of R has unique supremem and infimum

(ii) It is not necessary the supremum and infimum of a bounded subset X of R are the elements of X. They may be different from the elements of X.

(iii) For a non-empty bounded subset X of R, sup (x) $\geq$ inf (x)

EXAMPLE

1. Prove that the set of all positive real numbers is not bounded above.

Sol. :

Let r is upper bound of R^+ (if possible) so $1 \leq r$. Now we can say that $r > 0$. therefore $r + 1 > 0$ $r + 1$ is positive real number which presents that $r + 1 >$ upper bound r which is contradiction because for an upper bound $r_1 \leq r \ \forall \ r_1 \in R^+$.

Thus r is not upper bound. Hence set of + ve real numbers is not bounded above.

We can show that set of all negative real numbers is not bounded below. For this we consider s is lower bound of R^- (if possible). Then $s \leq -1$. Therefore $s - 1 < 0$, $s - 1$ is negative real number we can say that $s - 1 < s$ (lower bound of R^-) which is contradiction because for a lower bound $s_1 \geq s \ \forall \ s_1 \in R^-$. Thus s is not lower bound. Hence set of – ve real numbers is not bounded below.

2. Prove that for a set supremum and infimum (if exist) are unique ?

Sol. :

We consider r_1 and r_2 are supremum of set X . Since r_1 and r_2 are supremum so these are upperbound of X.

if r_1 is sup. then $r_1 \leq r_2$ (r_2 is upper bound) ...(1)

if r_2 is sup. then $r_2 \leq r_1$ (r_1 is upper bound) ...(2)

from (1) and (2)

$r_1 = r_2$

Hence supremum is unique.

We consider s_1 & s_2 are infimum of set X. then s_1 & s_2 are lower bound of set X.

If s_1 is infimum then $s_1 \geq s_2$ (lower bound) ...(3)

If s_2 is infimum then $s_2 \geq s_1$ (lower bound) ...(4)

From (3) & (4)

$s_1 = s_2$

3. Determine the supremum and infimum of the set { 5 } ?

Sol. :

The upper bound of the set { 5 } are 5 and all the real numbers which are greater than 5. The minimum number in all of these upper bound is 5. So 5 is the supremum. Similarly the lower bound of the set { 5 } are 5 and all the real numbers which are less than 5. The maximum number all of these lower bound is 5. So 5 is infimum of the set { 5 }. Hence 5 is supremum as well as infimum of the set { 5 }

4. If X is any subset of R s.t.

(i) X is bounded and non-empty

(ii) Sup x = Inf x

Then what can you say about X ?

Sol. :

Given x is non- empty and bounded subset of R and Sup X = Inf X

We take Inf X = Sup X = r

$\Rightarrow$ r is lower bound and upper bound of X.

$\Rightarrow r \geq r_1 \ \forall r_1 \in X$ ...(1)

& $r \geq r_1 \ \forall r_1 \in X$ ...(2)

From (1) & (2) $r_1 = r \ \forall r_1 \in X$

$\Rightarrow$ r is the only element belongs to X.

Thus we can say that X is singleton set { r }

5. Find supremum and Infimum of the set

$x = \{ x \in I \mid x^2 \leq 36\}$

Sol. :

$x = \{ x \in I \mid x^2 \leq 36\}$

or $x = \{ 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}$

which is finite subset of set of real numbers from this set we observe that the minimum number is – 6. The real numbers which are less than – 6 and – 6 are lower bound of X. The maximum number in all of these lower bound is – 6. Thus – 6 is the greatest lower bound of X i.e. – 6 is the infimum of X. The maximum number in the set x is + 6. The real numbers which are greater than 6 and 6 are upper bound of set x. The minimum number in all of these upper bound is + 6. So + 6 is the least upper bound i.e. it is the supremum of the set X.

Hence Inf X = – 6 , Sup X = + 6.

6. Write the supremum and infimum for the set x = { 1, 3, 5, 7, 9}

Sol. :

9 is the upper bound for x because $\forall$ x $\in$ x , x $\leq$ 9. Any number $x_1 < 9$ is not upper bound of x. So 9 is the supremum for x . 1 is the lower bound of set x because $\forall$ x $\in$ x , 1 $\leq$ x. Any no. $x_2 > 1$ is not lower bound of x so 1 is the infimum of the set x.

7. Is

(i) Every infinite set unbounded ?

(ii) Every subset for an unbounded set is unbounded?

Sol. :

(i) No. Every infinite set is not unbounded we consider the example $x = \left\{ \frac{1}{n} | n \in N \right\}$

the upper bound of this set is 1 and lower bound of this set is 0.

(ii) No. If we take set of integers I = {0, ± 1, ± 2,......}. For this set we consider the subset x = {0, ± 1, ± 2,......} which is bounded because – 2 is the lower bound and + 2 is the upper bound of this set x.

8. Is there exist a bounded set which

(i) has supremum but not infimum?

(ii) has infimum but not supremum?

Sol. :

(i) Yes, there exist infinity many sets.

one is x = { x $\in$ R | 3 < x $\leq$ 4}

The supremum of this set is 4 $\in$ x but infimum 3 $\notin$ x.

(ii) Yes, there exist infinitely many set S.

One is x = { x $\in$ R | 3 $\leq$ x < 4}

The infimum is 3 which is the element of x but supremum 4 is not belongs to this set.

9. Determine bound (if exist) of the following sets :

(i) $x = \left\{ \frac{1}{n} | n \in N \right\}$

(ii) $x = \left\{1 + \dfrac{(-1)^n}{\text{n}} \mid \text{n} \in \text{N}\right\}$

(iii) $x = \left\{1 + \dfrac{1}{\text{n}} \mid \text{n} \in \text{N}\right\}$

(iv) $x = \left\{1 - \dfrac{1}{\text{n}} \mid \text{n} \in \text{N}\right\}$

(v) Set of all negative integers.

Sol. :

(i) $x = \left\{\dfrac{1}{n} \middle| n \in N\right\}$

$1 \in$ x is the upper bound of x because $1 \geq$ x $\forall$ x $\in$ x . Any number which is less than 1 can not be upper bound of x. So 1 is the supremum of x. Since $\forall$ x $\in$ x, x ≥ 0 so 0 is the lower bound of x. Now we take an arbitrary vary small quantity m. Then $\exists$ a n $\in$ N s.t. $\dfrac{1}{\text{n}} \langle$ m . so m is not lower bound of x. Thus we can say that + ve real number greater than 0 cannot be lower bound of x. Hence 0 is the infimum of x.

Sup (x) = 1 & Inf (x) = 0

(ii) $x = \left\{1 + \dfrac{(-1)^n}{\text{n}} \mid \text{n} \in \text{N}\right\}$

We write the set x in tabular from by putting n = 1, 2, 3,.....

$$x = \left\{0\,,\ \frac{3}{2}\,,\ \frac{2}{3}\,,\ \frac{5}{4}\,,\ \frac{4}{5},......\right\}$$

$$= \{0\} \cup \left\{\frac{2}{3}\,,\ \frac{4}{5}\,,......,\ \frac{2\text{n}}{2\text{n} + 1}\,,\ ......\right\}$$

$$\cup \left\{\frac{3}{2}\,,\ \frac{5}{4}\,,\ -\ \frac{2n+1}{n},.....\right\}$$

$$= \{0\} \cup \left\{\frac{2\text{n}}{2\text{n} + 1} \mid \text{n} \in \text{N}\right\} \cup \left\{\frac{2n+1}{n} \mid \text{n} \in \text{N}\right\}$$

$= \{ 0 \} \bigcup Y \bigcup Z$ Where $y = \left\{ \dfrac{2n}{2n + 1} \mid n \in N \right\}$

$$z = \left\{ \frac{2n+1}{n} \mid n \in N \right\}$$

Now, $y = \left\{ \dfrac{2n}{2n + 1} \mid n \in N \right\}$

$$= \left\{ 1 + \frac{1}{2n+1} \mid n \in N \right\}$$

$$= \left\{ 1 + \frac{1}{3}, 1 + \frac{1}{5}, 1 + \frac{1}{7}, ... \right\}$$

We see in this set as n increasing the element's of x decreasing and tending to 1, $1+\dfrac{1}{3}=\dfrac{4}{3}$ is the largest element of this set Y.

For the set $z = \left\{ \dfrac{2n + 1}{2n} \mid n \in N \right\}$

$$= \left\{ 1 + \frac{1}{2n} \mid n \in N \right\}$$

$$= \left\{ 1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{6}, ... \right\}$$

elements of z are decreasing and tending to 1.

The largest element of this set is $1 + \dfrac{1}{2} = \dfrac{3}{2}$

Since $X = Y \cup Z \cup \{ 0 \}$

So supremum of $x = \max\left(\dfrac{3}{2}, \dfrac{3}{4} \right) = 3/2$

Infimum of $x = 0$

(iii) $x = \left\{1 + \frac{1}{n} \mid n \in N\right\}$ We can write the set x as

$$x = \left\{1 + \frac{1}{2} = 2,\ 1 + \frac{1}{2},\ 1 + \frac{1}{3},\ 1 + \frac{1}{4}, ...\right\}$$

As we are seeing that the elements of x are decreasing so the maximum number in this set is 2. 2 is upper bound of x. No number x < 2 is not upper bound of x. So 2 is the supremum of this set.

1 is the lower bound of x . If we choose an arbitrary real number s > 1 s.t. s is very - very close to 1 then ∃ a natural number n S.t. $1 + \frac{1}{n} \langle s$. Thus s can not be lower bound of x.

So we can say that 1 is the greatest lower bound i.e.

infimum. Sup x = 2, Inf x = 1

(iv) $x = \left\{1 - \frac{1}{n} \mid n \in N\right\}$ We can write this set as

$x = \left\{1 - 1 = 0,\ 1 - \frac{1}{2},\ 1 - \frac{1}{3},\ 1 - \frac{1}{4}, ...\right\}$. from one sight the least element of this set is 0. So 0 is the lower bound of x. Any number x > 0 cannot be lower bound of x. So 0 is the infimum from the set we see that 1 is the upper bound of this set. It we choose an arbitrary real number s < 1 s.t. s is vary- vary close to 1 then ∃ n ∈N s.t. $1 - \frac{1}{n} \rangle s$ thus s cannot be upper bound of x. So 1 is supremum of x. Hence Sup x = 1 and Inf x = 0

(v) I^- = Set of all negative integer. we write

$I^- = \{-1, -2, -3, ...\}$

I^- is bounded above. $-1 \in I^-$ is an upper bound of I^-. Now $-1 \in I^-$. So any number less than -1 can not be upper bound of I^-. Therefore -1 is the supremum of I^-.

We cannot find any real number m s.t.

$m \leq x \ \forall \ x \in I^-$. The set I^- is not bounded below. So infimum for this set does not exist.

2.11. Completeness property for R :

Completeness for the set of numbers with respect to boundedness :

Any set P of numbers whose Every non-empty subsets which is bounded above has a member of P for its supremum is complete.

e.g. The set of integer I is complete.

Completeness Property For R :

Every bounded above non-empty set of real number has a supremum in R. It is also called supremum property of R.

Note : (i) R is complete

(ii) R is ordered Field

(iii) Q is not complete.

2.12. Complete Ordered Field :

Let P is any ordered Field. P is called completed ordered Field if it is complete. In other words we can say that P is complete if Every non-empty subset P_1 of P which is bounded above has member of P for its supremum.

EXAMPLE

1. R is complete ordered field.

Note : Q is not complete ordered field

Theorem 1:

Any non-empty bounded below subset of real numbers has an infimum.

Proof :

Consider a non-empty bounded below subset P of R.

W define a set

$Q = \{ q \mid q = -p, p \in P\}$

It is given that P is bounded below. So $\exists$ lower bound p_1 of P. So we can say that $p_1 \le p \;\forall\; p \in P$

Now, $p_1 \le p \Rightarrow -p_1 \ge -p \Rightarrow -p_1 \ge q \;\forall\; q \in Q$

$\Rightarrow -p_1$ is upper bound of Q.

$\Rightarrow$ Q is bounded above.

Thus Q has supremum say q_1 (by completness property)

Now, $q_1 = \text{Sup } Q$

$\Rightarrow q \le q_1 \;\forall\; q \in Q$

$\Rightarrow -p \le q_1$

$\Rightarrow p \ge -q_1 \;\forall\; q \in P$

$\Rightarrow -q_1$ is lower bound of P.

it t' is lower bound of P then $-t'$ is an upper bound to Q and so

$q_1 \le -t'$

$\Rightarrow -Q_1 \ge t'$

Hence $-q_1 = \inf P$

Theorem 2 :

The set of all rational numbers i.e. Q is not complete ordered field.

Proof :

Consider the set $P = \{ p \mid p \in Q^+ \ \& \ p^2 < 2\}$ P is non-empty because $\frac{1}{2} \in P$. 2 is the upper bound of P i.e. P is bounded above. Thus we can say that P is non- empty bounded above subset of Q. We shall prove that there does not exist any rational number x which is supremum of P.

When $x \le 0$. In this case x can not be supremum of P because each clement of P is positive.

When $x > 0$ & $0 < x^2 < 2$

Consider $$y = \frac{4 + 3x}{3 + 2x} \qquad (1)$$

$$\Rightarrow y^2 - 2 = \frac{x^2 - 2}{(2x + 3)^2} \qquad (2)$$

and $$y - x = \frac{2.(2 - x^2)}{(2x + 3)} \qquad (3)$$

$x \in Q^+$ so from (1) $y \in Q^+$. $x^2 < 2$ then from (2) $y^2 < 2$. From (3) we can say that $y > x$. Now we have $y \in Q^+$ & $y^2 < 2$ this implies $y \in P$. Since $y > x$ so x cannot be upper bound for P.

When $x > 0$ and $x^2 = 2$. Since there does not exist any rational number whose square is 2. So this case is not possible.

When $x > 0$ and $x^2 > 2$

From (1) $$y = \frac{4 + 3x}{3 + 2x}$$

$x^2 > 2$ we can say from (1) , (2) & (3) $y \in Q^+$ s.t. $y^2 > 2$ & $y < x$ which implies $2 < y^2 < x^2$.

We take y_1 as arbitrary element of P.

Then,

$0 < y_1^2 < 2 < y^2 < x^2$ or $0 < y_1 < y < x$

$\Rightarrow$ x and y both are upper bound of P and x cannot be best upper bound of P because y which is less than x is upper bound of P. Set Q of rational number does not satisfy order completeness property.

Hence Q of rational number is not order complete.

Theorem 3 :

Natural number set N is not bounded above.

Proof :

We consider N is bounded above. Since N is bounded above and $N \neq \phi$ so N must have leas upper bound M (By order completeness property).

So $n \leq M \ \forall \ n \in N$

Since n + 1 is a natural number so

$n + 1 \leq M \ \forall \ n$

$\Rightarrow n \leq M - 1 \ \forall \ n$

$\Rightarrow M - 1$ is upper bound of N.

Thus we have an upper bound $M - 1$ of N which is less than the supremum of N. This is contradiction.

From above we can say that N is not bounded above.

2.13. Archimedean Property for real numbers :

Theorem 1 :

For $x \in R$ and $y \in R^+$ there exist a positive integer n s.t.

$n y > x$

Proof :

When $x \leq 0$ then theorem is obvious consider $x > 0$.

let we cannot find any $n \in I^+$ s.t. $ny > x$.

So $\forall \ n \in N$ we have $n y \leq x$

$\Rightarrow$ x is upper bound of the set

$P = \{ n y / n \in N\}$

Since $P \neq \phi$ and bounded above so

P must have least upper bound M (By completeness property)

So, $n y \leq M \ \forall \ n \in N$

$\Rightarrow (n + 1) y \leq M \ \forall \ n \in N$

$\Rightarrow n y + y \leq M$

$\Rightarrow n y \leq M - y \ \forall \ n \in N$

$\Rightarrow M - y$ is upper bound of P.

Thus we have $M - y$ which is upper bound of P is less than supremum M. Which is not possible so our assumption is wrong.

Thus $\exists$ some $n \in I^+$ s.t. $n y > x$.

2.14. Archimedean ordered field :

Let F be any ordered field. if for all $x, y \in F$ and $y > 0$ $\exists$ some $n \in I^+$ s.t. $n y > x$ then F is called Archimedean ordered field.

EXAMPLE

Real number Field R is an Archimedean field.

Corollary :

1. If $x \in R$ then $\exists$ a + ve integer n s.t. $n > x$.

2. If $x \in R^+$ then $\exists$ a + ve integer n s.t. $\frac{1}{n} \langle x$

Proof 1:

If we take y =1 in Archimedean property then we get the corollary 1.

Proof 2 :

In Archimedean property we take y = x & x = 1. we get

$n x > 1$

$$\Rightarrow x < \frac{1}{n}$$

Corollary :

3. Let $q \in R$ then $\exists$ two integer P and r s.t. $P < q < r$

Proof :

Given $q \in R$ we consider $q > 0$.

Now $1 \in R$ so by Archimedean property $\exists\, r \in N$ s.t. r. $1 > q$ or $q < r$.

We consider $q < 0$ and $1 \in R$ then by Archimedean property $\exists\, r \in N$ s.t. r. $1 > q$ i.e. $q > r$

Now we can say that in Every case by Archimedean property we can find $r \in N$ s.t.

$q < r$ ...(1)

When $q < 0$ then $-q$ is positive so (a + ve integer m s.t. $-q < m$ or $-m < q$

take $P = -m$

We have $P < q$ ...(2)

Now we have

$P < q < r$ from (1) & (2)

Example :

For Every positive real number x we can find a unique natural number n s.t.

$$\frac{n(n-1)}{2} \leq x \langle \frac{n(n+1)}{2}$$

Sol. :

Consider a real number $\left(2x + \frac{1}{4}\right)^{\frac{1}{2}} + \frac{1}{2}$

We can find a unique natural number n s.t.

$$n \le \left(2x + \frac{1}{4}\right)^{\frac{1}{2}} + \frac{1}{2} \langle n + 1$$

$$\text{or } \left(n - \frac{1}{2}\right) \le \left(2x + \frac{1}{4}\right)^{\frac{1}{2}} \langle n + \frac{1}{2}$$

$$\text{or} \left(n - \frac{1}{2}\right)^2 \le \left(2x + \frac{1}{4}\right) < \left(n + \frac{1}{2}\right)^2$$

$$\text{or, } n^2 + \frac{1}{4} - n \le 2x + \frac{1}{4} \langle n^2 + \frac{1}{4} + n$$

$$\text{or, } n^2 - n \le 2x \le n^2 + n$$

$$\text{or, } \frac{n(n - 1)}{2} \le x \langle \frac{n(n + 1)}{2}$$

2.15. Dedekind's Property For Real Numbers :

R be set of all real number and A & B are two non-empty set s.t. $A \cup B = R$ and Every element of A is less than element of B then we can find $P \in R$ s.t. $q < P \Rightarrow q \in A$ and $r > P \Rightarrow r \in B$

For Example :

We take $A = R^-$

and $B = R^+ \bigcup \{ 0 \}$

A and B are non-empty and $A \cup B = R$. Also we have Every element of A is less than B then $\exists\, 0 \in R$ s.t. $q < 0 \Rightarrow q \in A$ and $r > 0 \Rightarrow r \in B$.

Theorem 1

The order completeness implies and is implies by Dedekind's property.

Proof :

We consider two non-empty set A and B s.t. $A \cup B = R$. and each members of A is less than each member of B.

We prove that

Order completeness $\Rightarrow$ Dedekind's property.

i.e. we can find real number p s.t. $q < p$

$\Rightarrow q \in A$ and $r > p \Rightarrow r \in B$

Since A and B are non-empty so we can take $r \in B$. Since each member A is less than each member B so $q < r \ \forall \ q \in A$

$\Rightarrow$ r is upper bound for A

$\Rightarrow$ A has supremum in R [$\because$ By completeness property of R]

Consider Supremeum A = p

take $r > p$

We can say $r \notin A$

Since $r \notin A$ then $r \in B$ [$\because A \cup B = R$]

Thus we have $r > p \Rightarrow r \in B$

further consider $q > p$. So $\exists \ s \in A$ s.t. $q < s$

[because Sup A = p $\Rightarrow$ q is not upper bound for A]

$\because$ each member of A is less than each member of B so we have r $s \in A$ and $q > s$ this implies $q \notin B$ or $q \in A$.

Therefore we can find $p \in R$ s.t. $q < p \Rightarrow q \in A$ and $r > p \Rightarrow r \in B$.

Thus we can say that completeness implies Dedekind property. Consider a non-empty bounded above subset x of R. Take k_1 as upper bound of x.

Since k_1 is upper bound of $x \Rightarrow p \leq k_1 \ \forall \ p \in x$

Consider B is the set of all upper bound of x. So B contains k_1. We can say B is non-empty. If $A = R - B$ then $x \subset A$ and A is non-empty. $A \cup B = R$. Consider $q \in A$ and $r \in B$. Then q and r distinct. Now it $q > r$ then $r \in B$ implies $q \in B \Rightarrow A \cap B$ is non- empty which is not possible so we have $q < r$. By Dedekind's property we can find $p \in R$ s.t. $q < p \Rightarrow q \in A$ and $r > p \Rightarrow r \in B$. When $r > p$ then r is not belongs to A. So r is not the element of x [$\because x \subset A$]. Thus the real no.

$r > p$ is not belongs to x. Thus $q \leq p \ \forall \ q \in x$ i.e. p is upper bound for x.

Now consider $q_1 < p$.

$q_1 < p \Rightarrow q_1 \in A \Rightarrow q_1 \notin B$

So x is not bounded above by q_1. Now we have any real number $q_1 < p$ is not upper bound of x and p is upper bound for x. So p is supremum of x.

Thus Dedekind property implies completeness property.

2.16. Representation of Real numbers on a line :

We take a straight line. Consider point o on it. It divide the straight line in two parts. One is right from 0 and second is left from 0. Right part is positive and left part is negative. We take another point P on the positive part. o represents 0 and P represent's 1. OP is 1 unit. Each point on the line can be associated with exactly one real number. The point's in the positive part of line represents the positive real number i.e. positive real numbers are presents in the right hand side of o. Negative real numbers are presented in the left side of o. This line is called real line R.

2.17. Dedekind Cantor axioms :

Corresponding to Every real number there is unique point on the directed line and conversely corresponding to unique point on the directed line there is a unique real number.

Note : There is one to one correspondence between the real numbers and the points on directed line.

2.18. Denseness Property :

Theorem 2

Between two different real numbers there always lies a rational number and so infinity may rational numbers.

Proof :

P and r any two real number s.t. $p < r$ then $r - p > 0$. According to Archimedean property there exist a integer $n \in 1^+$ s.t. $n(r - p) > 1$ or $n r - n p > 1$ or $n r > 1 + n p$.

We can also find unique $m \in I$ s.t. $m > n p \geq m - 1$ so that $n p + 1 \geq m > n p$.

From above we have

$n r > n p + 1 \geq m > n p$

$$\Rightarrow r > \frac{m}{n} > p$$

Here $\frac{m}{n} \in Q$

or $p < q < r$ where $q = \frac{m}{n}$

Thus between two real number P and r $\exists$ a rational number q. continuing above procedure for P and q & q and r we get rational number q_1 and q_2 s.t.

$p < q_1 < q$ & $q < q_2 < r$

$\Rightarrow p < q_1 < q < q_2 < r$

Continuing in this way we get infinitely many rational numbers between two distance real numbers.

Theorem 3

Between two different real numbers there always lies an irrational number and so infinitely many irrational numbers.

Proof :

P and r two real number s.t. $r > p$ so $r - p > 0$. According to Archimedean property $\exists$ a + ve integer n s.t.

$n (r - p) > x$ where x is positive irrational number.

$$r - p \rangle \frac{x}{n} \Rightarrow r \rangle p + \frac{x}{n}$$

Since $p + \frac{x}{n} \rangle p + \frac{x}{2n} \rangle P$

So, $r \rangle p + \frac{x}{n} \rangle p + \frac{x}{2n} \rangle P$

Now, $\left(P + \frac{x}{n}\right) - \left(P + \frac{x}{2n}\right) = \frac{x}{2n}$ is irrational so at least

one of $P + \frac{x}{n}$ and $P + \frac{x}{2n}$ is irrational number. Take this irrational number q (say). Thus we have $r > q > p$ or $p < q < r$. Continuing above procedure for p and q & q and r we have irrational number q_1 and q_2 between p and q & q and r s.t. $p < q_1 < q < q_2 < r$. Continuing in this way we get infinity many irrational numbers between two distinct real number p and r.

Theorem 4

There always lie infinite real number between two distinct real numbers.

Proof :

The proof of this theorem is follows from one of the theorem 1 and 2.

Exercise 1(A)

1. Prove that $\sqrt{5}$ is not rational number ?

2. Show that if $x \in \,]\, a, b \,[$ then $\left|x - \frac{a+b}{2}\right| \langle \frac{a - b}{2}$

3. Show that $|x + y + z| \leq |x| + |y| + |z| \ \forall \ x, y, z \in R$?
4. Prove that $|x - y| = |y - x|$
5. Prove that there does not exist any rational number whose squall is 3?
6. If $x, y \in (a, b)$ then prove that $|x - y| < b - a$?
7. If $x \in R$ then prove that $|x| = \sqrt{x^2}$?
8. Show that $\max\{x, y\} = ½ (x + y + |x - y|)$ and $\min \{x , y\} = ½ \{x + y - |x - y|\} \ \forall \ x, y \in R$
9. Show that the set $x = \{x \mid x = 2^n, n \in N\}$ is bounded below ?
10. Find the supremum and infimum of the following sets :
    (i) $x = \{x \in I \mid x^2 \leq 4a\}$
    (ii) $x = \{o, 1, 2, 3, 4, 5, 6\}$
    (iii) $(3, 4)$
    (iv) $x = \{x \mid x = 2^n, n \in N\}$
    (v) $x = \left\{ \frac{3n + 2}{2n + 1} \middle| n \in N \right\}$
    (vi) $x = \left\{ \frac{1}{5n} \middle| n \in I - \{0\} \right\}$
    (vii) $x = \left\{ \frac{1}{n} - \frac{1}{m} \middle| n , m \in N \right\}$
    (viii) $x = \left\{ \pi + 1, \pi + \frac{1}{2}, \pi + \frac{1}{3}, ... \right\}$
    (ix) $x = \left\{ m + \frac{1}{n} \middle| m , n \in N \right\}$
11. Show that if $x = \{x \in R \mid x = n + 3, n \in N\}$, x is unbounded?
12. Show that set R of all real numbers is unbounded?
13. Show that for the set $x = \left\{ \pi + \frac{1}{2}, \pi + \frac{1}{4}, \pi + \frac{1}{8} \ .... \right\}$ The infimum is $\pi + \frac{1}{2}$
14. Find supremum and infimum if they exist :

(i) $x = \left\{ x \in Q \mid x = (-1)^n \left(\frac{1}{4} - \frac{4}{m} \right), n \in N \right\}$

(ii) $x = \left\{ x \in Q \mid x = \frac{(-1)^n}{n}, n \in N \right\}$

(iii) $]1, 2 \bigcup [3, 8[$

(iv) $x = \left\{ x \in Q \mid x = (-1)^n \left(\frac{1}{n} - \frac{4}{n} \right), n \in N \right\}$

(v) $x = \left\{ \left(1 - \frac{1}{n} \right) \sin \frac{n\pi}{2} \middle| n \in N \right\}$

15. Give an example of a set which is ordered field but not complete?
16. Show that x = 0 if $0 \le x \le \frac{1}{n} \; \forall n \in N$?
17. Show that x = 0 if $0 \le x < \in \;\; \forall \in > 0$?
18. Show that set Q of all rational numbers is an Archimedean ordered field?
19. If x is positive real number, then there exist a unique natural number n s.t.

$$\frac{n(n-1)(2n-1)}{6} \le x \langle \frac{n(n+1)(2n+1)}{6}$$

ANSWERS EXERCISE 1(A)

10. (i) Sup x = 7, infx = – 7
    (ii) Sup = 6, Inf = 0
    (ii) Inf = 3, Sup = 4
    (iv) Not bounded above, Inf = 2
    (v) Inf $= \frac{3}{2}$, Sup = $= \frac{5}{3}$
    (vi) Inf $= -\frac{1}{5}$, Sup = $= \frac{1}{5}$
    (vii) Inf = – 1, Sup = 1

(viii) Sup = $\pi + 1$, Inf = π

(ix) Inf = 1, Sup does not exist.

14. (i) Inf = $-\frac{7}{4}$, Sup = 3

(ii) Inf = – 1 , Sup = ½

(iii) Inf = 1, Sup = 8

(iv) Inf = $-\frac{3}{2}$, Sup = 3

(v) Inf = – 1, Sup = 3

2.19. Neighborhood of a point :

Let R be a set of all real numbers. S is any non-empty subset of R. 'a' is any real number. Then S is said to be Neighborhood of a if $\exists \in > 0$ s.t.

$a \in (a - \in, a + \in) \subset S$.

Or we say that $\exists$ an open interval I which contains a and is also contained in S.

Note : We use Nhd. In place of Neighborhood.

2.20.Deleted Neighborhood :

Let $a \in R$ and S is Nhd. of a then the set $S - \{a\}$ is called deleted Nhd. of a.

EXAMPLE

1. Find the Nhd. of 5 in the following sets :

(i)] 2, 6 [(ii) [2 , 6 [

(iii) [2 , 6] (iv)] 5 , 7]

(v)] 5 , 7 [

Sol. :

(i) Since $\exists$ open interval] 3 , 6 [$\subset$] 2 , 6 [and 5 $\in$] 3 , 6 [so] 2, 6 [is a Nhd. of 5.

(ii) Since 5 $\in$] 2, 6 [$\subset$ [2, 6 [so [2 , 6 [is a Nhd. of 5.

(iii) Since 5 $\in$] 2, 6 [$\subset$ [2, 6] so [2, 6] is a Nhd. of 5.

(iv) Since 5 $\notin$] 5 , 7] so it is not Nhd. of 5.

(v) Since 5 $\notin$] 5 , 7 [so] 5 , 7 [is not a Nhd. of 5.

2. Show that Every open interval is a Nhd. of each of its points.

Sol. :

] p, q [any open interval and r is any arbitrary point of] p, q [i.e. $p < r < q$. We consider $\in$ M the minimum of $r - p$ and $q - r$ then $r \in (r - \in, r + \in) \subset$] p, q [. Thus we can say that] p , q [is a Nhd. of r. Hence Every open interval is a Nhd. of each of its points.

3. Every loosed interval [p, q] is a Nhd. of each of its points except initial point P and final point q.

Consider open interval (p, q) and r is an arbitrary point of (p, q) then we have already Prove that Every open interval is a Nhd. of each of its points. So (p, q) is a Nhd. of r. we have $(p, q) \subset [p, q]$. We can write $r \in (p, q) \subset [p, q]$ so [p, q] is Nhd. of r or [p, q] is a Nhd. of Every element of (p, q). Now for $\in > 0$ we have $p \in (p - \in, p + \in)$ but $(p - \in, p + \in) \not\subset [p - q]$

Thus $\in > 0$ s.t. $(p - \in, p + \in) \subset [p, q]$ So [p , q] is not Nhd. of P. Similarly $\in > 0$ s.t. $q \in (q - \in, q + \in) \not\subset [p, q]$ so [p, q] is not Nhd. of q.

Hence Every closed interval is Nhd. of each of its points except starting and final point.

4. Show that set Q of all rational numbers is not Nhd. of any rational number.

Sol. :

Let $r \in Q$ is any arbitrary rational number if take any $\in > 0$ than we can not find the open interval $(r - \in, r + \in)$ s.t. $(r - \in, r + \in) \subset Q$ because in $(r - \in, r + \in)$ there are infinitely irrational numbers (Denseness property) Hence Q is not Nhd. of any rational number.

5. Prove that if x is any non- empty finite then it is not Nhd. of each of its points ?

Sol. :

It is given that x is any non-empty finite set. Let $x \in x$ is any arbitrary point of x. Then for any $\in > 0$ we can not find the open interval $I = (x - \in, x + \in)$ s.t. $I \subset x$. because I contains infinite points distinct from the points of x. Thus Every $I = (x - \in, x + \in) \not\subset x$. Hence x is not Nhd. of each of its point.

6. Prove that the set R of all real numbers is a Nhd. of each of its points?

Sol. :

Let $r \in R$ is any arbitrary real number. Then for any $\in > 0$ we have $r \in (r - \in, r + \in)$ and $(r - \in, r + \in) \subset R$ i.e. $r \in (r - \in, r + \in) \subset R$.

So, R is Nhd. of each of its points.

7. Show that I^+ set i.e. set of all positive integer is not Nhd. of any positive integer ?

Sol. :

Given I^+ is a set of all positive integer $r \in I^+$ any arbitrary positive integer. For any $\in > 0$ we can not find the open interval $(r - \in, r + \in)$ s.t. $(r - \in, r + \in) \not\subset I^+$ because in $(r - \in,$

$r + \in$) there are infinite real numbers which are not positive integers. So I^+ is not Nhd. of any positive integer.

Some theorems on Neighborhood :

Theorem 1:

$a \in S$ if S is a Nhd. of $a \in R$.

Proof :

Given $a \in R$ any real number and S is Nhd. of a than $\exists \in > 0$ s.t.

$a \in (a - \in, a + \in) \subset S$

$\Rightarrow a \in S$

Theorem 2:

Every $a \in R$ has at least one Nhd.

Proof :

Take $a \in R$ any arbitrary. Then for any $\in > 0$

We have $a \in (a - \in , a + \in) \subset R$

R is always Nhd. of a.

Hence Every $a \in R$ has at least one Nhd.

Theorem 3:

If for any point P , N_1 and N_2 are two Nhds. then $N_1 \cap N_2$ is also Nhd. of P.

Proof :

Since N_1 is Nhd. of P so $\exists \in_1 > 0$ s.t.

$(P - \in_1 , P + \in_1) \subset N_1$

Similarly N_2 is Nhd. of P so $\exists \in_2 > 0$ s.t.

$(P - \in_2 , P + \in_2) \subset N_2$

Now take $\in = \min (\in_1 , \in_2)$

Then $(P - \in , P + \in) \subset (P - \in_1 , P + \in_1) \subset N_1$

and $(P - \in , P + \in) \subset (P - \in_2 , P + \in_2) \subset N_2$

So, $(P - \in , P + \in) \subset N_1 \cap N_2$

Hence $N_1 \cap N_2$ is Nhd. of P.

Theorem 4:

If X is a superset of any Nhd. N of point a then x is also Nhd. of a.

Proof :

Since N is Nhd. of a so $\exists \in > 0$ s.t.

$a \in (a - \in, a + \in) \subset N$

Since, $X \supset N$ so we have

$a \in (a - \in, a + \in) \subset N \subset x$

$\Rightarrow a \in (a - \in, a + \in) \subset X$

Hence x is Nhd. of a.

Theorem 5:

The necessary and sufficient condition for a non-empty subset S of R. is a Nhd. of $P \in R$ is that $\exists\, n \in > I^+$ s.t.

$$\left(P - \frac{1}{n}, P + \frac{1}{n}\right) \subset S$$

Proof :

Necessary Part

It is given that S is non-empty subset of R and Nhd. of $P \in R$.

We have to Prove $\exists\ n \in I^+$ s.t. $\left(P - \frac{1}{n}, P + \frac{1}{n}\right) \subset S$

Since S is Nhd. of P so $\exists \in > 0$ s.t.

$p \in (p - \in, p + \in) \subset S$

For $\in > 0$ we can take $n \in > I^+$ s.t. $\frac{1}{n} < \in$

$$\frac{1}{n} \langle \in \Rightarrow P + \frac{1}{n} \langle P + \in \qquad ...(1)$$

and $\frac{1}{n} \langle \in \Rightarrow -\frac{1}{n} \rangle - \in$

$$\Rightarrow p - \in \langle p - \frac{1}{n} \qquad ...(2)$$

From (1) & (2)

$$\left(p - \frac{1}{n}, p + \frac{1}{n}\right) \subset (p - \in, p + \in)$$

Thus $\exists\ n \in\ > I^+$ s.t. $\left(p - \frac{1}{n}, p + \frac{1}{n}\right) \subset S$

Sufficient Part :

Given $\exists\ n \in\ > I^+$ s.t.

$$\left(p - \frac{1}{n}, p + \frac{1}{n}\right) \subset S$$

to prove S to Nhd. of P

Since $\left(p - \frac{1}{n}, p + \frac{1}{n}\right)$ is such open interval which contains P and is contained in S

therefore S is Nhd. of P.

2.21. Adherent Point :

Let $a \in R$ be any point. Then a is called the adherent point of set $x \subset R$ if Every Nhd. of P contains a point x. We denote the set of all a adherent points of x as Adh x and read as adherence of A.

By definition of Adherent point we can say $x \subset$ Adh x.

2.22. Limit Point (or accumulation point or cluster point or condensation point) :

Let $a \in R$ be any real number. Then a is said to be limit point of $x \subset R$ if Every Nhd. of a has a point of x distinct From a.

Or we can say that a is said to be limit point of $x \subset R$ if and only if $\forall$ Nhd. s of a, $(s \cap x) - \{a\} \neq \phi$

Or $\forall$ Nhd. s of a, $(s - \{a\}) \cap x \neq \phi$

Or $\forall\ \in > 0$ (] $a \in$ ($a - \in, a + \in$[$\cap$ x) - $\{a\} = \phi$

Note :

1. It is clear from definition that Every limit point of x is adherent point of x. Converse is not.

   For Example :

   We consider the set $x = \left\{\frac{1}{n} \middle| n \in N\right\}$ 1 is the adherent point but it is not the limit point.

2. It is not necessary that the limit point belongs to x.
3. The set of all limit points is called the Derived set. If we take the derived set of x then we write D (x).
4. The points $x \in X$ which are not limit point of x are called the isolated point.

5. Each point of set x is either limit point or isolated point.

Theorem 1 :

For any subsets x and y of R

(i) $D(\phi) = \phi$

(ii) If $X \subset Y$ then $D(X) \subset D(Y)$

(iii) $D(X \cap Y) \subset D(X) \cap D(Y)$

(iv) $D(X \cup Y) = D(x) \cup D(Y)$

(i) For $P \in R$, R is Nhd. of P and $R \cap \phi = \phi$ So we can say.

$x \notin$ limit point of ϕ any real number is not limit point of ϕ. Hence $D(\phi) = \phi$

(ii) To Prove It $X \subset Y$ then $D(x) \subset D(Y)$ take P as a limit point of X i.e. $P \in D(X)$. So this implies Every Nhd. N of P contains a point of X distinct from P. Since $X \subset Y$ so Every Nhd. N of P contains a point of Y distinct from a this implies $a \in D(Y)$. Thus we have $X \subset Y \Rightarrow D(X) \subset D(Y)$

(iii) To Prove if $D(X \cap Y) \subset D(X) \cap D(Y)$ Since $X \cap Y \subset X$ and Y

Then from (ii) $D(X \cap Y) \subset D(X)$ and $\subset D(Y)$

Thus we have $D(X \cap Y) \subset D(X) \cap D(Y)$

(iv) W have to prove $D(X \cup Y) = D(x) \cup D(Y)$

For this we shall show

$D(X \cup Y) \subset D(x) \cup D(Y)$

and then $D(X) \cup D(Y) \subset D(x \cup Y)$

Since X , $Y \subset X \cup Y$ So By (ii) We have

$D(x), D(Y) \subset D(X \cup Y)$ or $D(x) \cup D(y) \subset D(X \cup Y)$ ...(1)

Let P be a limit point of $X \cup Y$. Consider P is not belongs to $D(X) \cup D(Y)$. Therefore P is not belongs to D(X) and is not belongs to D(Y) i.e. P is not limit point of X and not of Y or we can say that we can find a Nhd. N_1 of P which contains no point of X distinct from P and a Nhd. N_2 of P which contains no point of Y distinct from P. This mean $N_1 \cap N_2$ Nhd. of P has not any point of X and Y distinct from P i.e. $N_1 \cap N_2$ does not have any point of $X \cup Y$ distinct from P i.e. $P \notin D(X \cup Y)$.So we have $P \in D(x) \bigcup D(y) \Rightarrow P \notin D(X \cup Y)$ Thus

$D(X \cup Y) \subset D(x) \cup D(Y)$ ...(2)

From (1) & (2) we have.

$D(X \cup Y) = D(x) \cup D(Y)$

EXAMPLE

1. Show that D(R) = R ?

Sol. :

Let $a \in R$ be any arbitrary real number. Then $(a - \in, a + \in)$ has infinite real number so we can say that $\forall \in > 0$ the open interval $(a - \in, a + \in)$ has atleast one real number distinct from a. Thus all real numbers are limit point of R. Hence D(R) = R.

2. Show that D (R – Q) = R

Sol. :

Consider $a \in R$ be any arbitrary real number. Then $\forall \in > 0$ the interval $(a - \in, a + \in)$ Contains infinity many irrational numbers. (Denseness property) Distinct from a since a is arbitrary so we have D(R – Q) = R

3. Find D(x) where $X = \left\{ \frac{1}{m} + \frac{1}{n} \,\middle|\, m, n \in N \right\}$ **?**

Sol. :

We have $X = \left\{ \frac{1}{m} + \frac{1}{n} \,\middle|\, m, n \in N \right\}$ first take m fix and n vary.

Then as $n \to \infty$ then $\frac{1}{m} + \frac{1}{n} \to \frac{1}{m}$ (m is fixed). Thus X has limit point $\frac{1}{m}$.

As $m \in N$ so the all points of the set $\left\{ \frac{1}{m} \,\middle|\, m \in N \right\}$ are limit points of X.

Now we take m & n both vary. as $m \to \infty$ and $n \to \infty$ then $\frac{1}{m} + \frac{1}{n} \to \infty$.

Thus we can say that 0 is the limit point of x.

Thus we have $D(X) = \left\{ \frac{1}{m} \,\middle|\, m \in N \right\} \cup \{0\}$ ...(1)

We have n fixed and m vary. As $m \to \infty$ then $\frac{1}{m} + \frac{1}{n} \to \infty$

Thus $\frac{1}{n}$ is the limit point of x i.e. all the point of $\left\{ \frac{1}{n} \,\middle|\, n \in N \right\}$

are limit point of X. Thus $D(X) = \left\{ \frac{1}{n} \middle| n \in N \right\} \cup \{0\}$ ...(2)

both (1) & (2) are same.

Now $D^2 (x) = \{0\}$

and $D^3 (x) = \phi$

Thus we can say that X is first species and of second order.

4. Find the limit point of the set $X = \left\{ P + \frac{1}{n} \middle| n \in N \right\}$ **?**

Sol. :

Consider $x \in R$ be any real number. Then by Trichotomy law exactly one of the following is true.

$p < x$, $p = x$, $p > x$

If $p < x$ then we can find $M \in I^+$ s.t.

$$\Rightarrow M \le \frac{1}{x-p} < M+1$$

$$\Rightarrow \frac{1}{M} \ge x - p \langle \frac{1}{M+1}$$

$$\Rightarrow P + \frac{1}{M} \ge x \rangle p + \frac{1}{M+1}$$

Thus the Nhd. $\left(P + \frac{1}{M+1} , p + \frac{1}{M}\right)$ of x contains no point of x distinct from x. So $x > p$ can not be limit point of P.

If $p = x$ then we can find a + ve integer $M \in I^+$ s.t. $\in \rangle \frac{1}{n} \forall n \ge M$ Thus all Nhd. $(p - \in , p + \in)$ has infinitely many points of x because $p+\frac{1}{n} \in (p - \in , p + \in) \forall n \ge m$.

If $p > x$ then we have $x-1 < p < p+\frac{1}{n} \forall n$. Thus the Nhd. $(x - 1, p)$ has no point of x. Thus we can say that $x < p$ can not be limit point of x.

Hence x has one limit point which is p.

5. Find D (Q) ?

Sol. :

Let $a \in R$. The $\forall \in > 0$, the open interval $(a - \in , a + \in)$ contains infinity many rational numbers (Denseness property). Thus the interval $(a - \in , a + \in)$ contains infinitely points of Q distinct from a. So a be limit point of Q. It a is arbitrary then Every real number is limit point of Q. Hence D(Q) = R.

6. Show that D(I) = ϕ, where I is the set of all integers?

Sol. :

Let $a \in I$ be any integer. We consider $\in > 0$ s.t. $\in = \frac{1}{2}$. Then the interval $\left(a - \frac{1}{2}, a + \frac{1}{2}\right)$ contains no integer distinct from a thus we can say that any $a \in I$ is not limit point of I.

If $a \notin I$. Then we can find integer p s.t. $p < a < p + 1$. So the interval $(p, p + 1)$ which is Nhd. of a does not contain any integer distinct from a. Thus a is not limit point of I. Hence $D(I) = \phi$.

7. Show that D (0 , 1) = [0, 1]

Sol. : Let $a \in [0, 1]$ be any arbitrary number in [0, 1] . Then the interval $(a - \in , a + \in)$ contains infinitely many points of]0, 1[distinct from a. Thus a is limit point of (0, 1)

If $a \notin [0 , 1]$. Then we consider $\in > 0$ s.t.

$\in < |a - 0|$ and $\in < |a - 1|$. The interval $(a - \in , a + \in)$ has no point of (0, 1). So a is not limit point of (0, 1)

Thus from above we can say that D (0, 1) = [0, 1]

Bolzano- Weierstrass Theorem 1 :

If X is infinite bounded set of real numbers then x has a limit point.

Proof :

It is given x is infinite bounded set of real numbers. Let m and M are bound of x. Let we define a new set S s.t.

$S = \{x \in R \mid$ number of elements which are belongs to X and less than x is finite$\}$

From S we can say that m belong to S. Therefore $S \neq \phi$. M is upper bound of S. Since S is non-empty and bounded above then by order completeness property S has supremum. Suppose M_1 is supremum of S and $(M_1 - \in, M_1 + \in)$ is Nhd. of M_1. Now we have M_1 is supremum of S. So we can find at least one point x of S s.t. $M_1 - \in < x$. $x \in S$ so at most finite number of points of X less than x and $M_1 - \in$ exceed finite number element of X almost. $M_1 + \in \notin S$ since M_1 is supremum of S. So there are infinitely many points of x less than $M_1 + \in$ from above we can say that the Nhd. $(M_1 - \in, M_1 + \in)$ has infinitely many elements of x. Hence M_1 is the limit point of X.

1. **Show that the set** $X = \left\{2 + \frac{1}{n} | n \in N\right\}$ **has a limit point?**

Sol. :

The set $X = \left\{2 + \frac{1}{n} | n \in N\right\}$ has infinitely many points. It has 3 and 0 as upper and lower bound respectively. So x is bounded set. Thus x is infinitely bounded set of real numbers. Hence by Bolzano-weierstrass theorem x has a limit point.

2. **Prove that set** $X = \left\{\frac{1}{n} | n \in N\right\}$ **has a limit point ?**

Sol. :

Since $X = \left\{\frac{1}{n} | n \in N\right\}$ is infinitely bounded set (lower bound 0 and upper bound 1). So by Bolzano-Weierstrass theorem it has a limit point.

2.23. Countable set :

Any set X is said to be countable set if it is either finite or denumerable. A set X is said to be denumerable (or countable infinite) if $\exists$ a mapping $f : N \to x$ which is $1 - 1$ and onto. The set which neither finite nor denumerable is called uncountable set.

1. **We consider the set** $X = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\right\}$. There exist a mapping between X and N which is $1 - 1$ and onto i.e.

   $f : N \to X$ s.t.

   $$f(n) = \frac{n}{n+1} \quad \forall\, n \in N$$

2. **The set x = { 2, 4, 6,.....} is countable because $\exists$ 1 – 1 onto map $f : N \to x$ s.t. f(x) = 2n**
3. **The set x = { 1, 3, 5, ...} is countable because it is denumerable i.e. $\exists$ 1 – 1 onto map $f : N \to x$ s.t. f(x) = 2n –1.**
4. **Show that set I of all integers is countable ?**

Sol. :

We consider a mapping

$f : N \to I$ s.t.

$$f(n) = \begin{cases} \dfrac{n-1}{2} & , \text{ When n is odd} \\ \dfrac{-n}{2} & , \text{ When n is even} \end{cases}$$

Now we shall show f is 1 – 1 and onto.

$f(1) = 0$

$f(2) = -1$

$f(3) = 1$

$f(4) = -2$

$f(5) = 2$

$f(6) = -3$

It is easily seen that different element of N have different image in I. So f is 1 –1.

Now let $m \in I^+ \cup \{0\} \Rightarrow 2m \in I^+ \cup \{0\} \Rightarrow 2m + 1 \in I^+$

i.e. 2m + 1 is a rational odd number so

$$f(2m + 1) = \frac{(2m + 1) - 1}{2} = m$$

Thus m has the preimage 2m + 1 in N. We let $m \in I^-$. $m \in I^- \Rightarrow -2m \in I^+$ i.e – 2m is even Natural number or even positive integer. $f(-2m) = \dfrac{-(-2m)}{2} = m$

m has preimage –2m.

Thus each element of I has preimage in N. So f is onto. Hence set of all integers is countable

2.24. Some theorem on countability :

Theorem 1:

Let x be any countable set then Every subset of X is countable.

Proof :

We consider $y \subset X$.

If y is finite then it is countable. Now let X is denumerable set and y is infinite. Since X is denumerable set so X can be written as $<x_1, x_2, \ldots\ldots x_n \ldots\ldots>$ infinite sequence. We take n_1

is the smallest natural number s.t. $x_{n_1} \in y$ and further we take n_2 is smallest natural number s.t.

$n_2 > n_1$ and so on. Thus we have

$y = \{x_{n_1}, x_{n_2}, x_{n_3} ......\}$. Clearly the function $f : N \to y$ defined by $f(r) = x_{n_r}$ is bijective. Hence y is denumerable i.e. countable.

Theorem 2:

If X is uncountable set then Every superset of X is uncountable.

Proof :

Let $y \supset x$. If y is countable then theorem 1 says x is also countable which is not possible because x is uncountable. Hence Every superset of uncountable set is uncountable.

Theorem 3:

If x and y are countable sets then $x \cup y$ is also countable.

Proof :

Given x, y and two countable set. To prove $x \cup y$ is also countable. Let the elements of x and y is arranged in definite order. So take $x = \{a_1, a_2......\}$

$y = \{b_1, b_2 .....\}$

Take x & y have no common elements. Then $x \cup y = \{C_1, C_2,...\}$

Here $C_{2n} = b_n$ & $C_{2n-1} = a_n$, $n \in N$

Thus each element of $x \cup y$ has definite place in the above arrangement. So $x \cup y$ is countable.

Note 1: If $x_1, x_2.....,x_n$ are finite number of countable sets then $\bigcup_{t=1}^{n} x_i$ is also countable.

Note 1: The union of countable family of countable sets is also countable.

Note 1: Intersection of two countable sets is countable set.

EXAMPLE

5. Show that set Q of all rational numbers is countable?

Sol. :

We can write $Q = \bigcup_{n \in N} x_n$

Where $x_n = \left\{\frac{0}{n}, \frac{1}{n}, -\frac{1}{n}, \frac{2}{n}, \frac{-2}{n}, ...\right\}$

Now, We take the function

$f : N \to x_n$ defined by

$$f(k) = \begin{cases} \dfrac{-k}{2n} & , \quad \text{k is even} \\ \dfrac{k-1}{2n} & , \quad \text{k is odd} \end{cases}$$

f is bijective function i.e. f is 1 – 1 onto. So we can say that x_n is countable. Now Q is the $\bigcup_{n \in N} x_n$. We know that the union of countable collection of countable sets is also countable so s is also countable.

So Q in countable set.

Note : Set Q^+ is countable.

6. Show that the set R of all real numbers is uncountable ?

Sol. :

We have to prove set R of all real numbers is not countable. If possible let it is countable. We know that Every subset of countable set is countable. If we take a subset of R which is not countable then our assumption is wrong i.e. we have already assumed that R is countable which is wrong by showing a subset of R which is not countable. For this we take the set closed interval [0, 1]. If possible we take [0,1] is countable. Since [0,1] is countable then it is finite or denumerable.[0,1] is not finite so it is denumerable. This implies there is enumeration α_1, α_2 ..... of elements of closed interval [0,1]

We write

$\alpha_1 = 0 . \beta_{11}\ \beta_{12}\ \beta_{13} ...... \beta_{1n}$

$\alpha_2 = 0 . \beta_{21}\ \beta_{22}\ \beta_{23} ...... \beta_{2n}$

....

....

....

$\alpha_n = 0 . \beta_{n1}\ \beta_{n2}\ \beta_{n3} ...... \beta_{nn}$

Where all β_{ij} are belong to the set $\{n \in I \mid 0 \le n \le 9\}$

Now we take a no. r, s.t; r can be expressed in decimal representation as $\gamma = 0\ \gamma_1 \gamma_2 .... \gamma_{n...}$

Here for all i = 1, 2,......., n, ..., $\gamma_i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\gamma_i \neq \beta_{ij}$. Obviously γ is the element of [0,1] and $\gamma \neq \alpha_n\ \forall\ n$ Now we can say γ is not involved in enumeration. Thus we get contradiction. Therefore [0,1] is not countable. Now we have show that a subset [0,1] of R is not countable. Thus our assumption is wrong i.e. R is countable is wrong. Hence R is uncountable.

7. Show that the set of all irrational numbers is uncountable?

Sol. :

R be the set of all real numbers and Q set of all rational numbers. We consider set of all irrational numbers is countable.

Now R = (R – Q) $\cup$ R $\Rightarrow$ R is countable.

[$\because$ R is union of two countable set] Which is contradiction. So R – Q i.e. set of all irrational numbers is uncountable.

Exercise 1(B)

1. Give an example of the following sets :
   (i) A set which is Nhd. of any of its points?
   (ii) A set which is Nhd. of each of its points?
2. Show that]0,1[is Nhd. of ½ ?
3. Is the set x =]2, 3[$\cup$]5, 6[is Nhd. of $\frac{5}{2}$? Justify your answers?
4. Prove that the close interval [5, 7] is Nhd. of 6 but not 5 and 7?
5. Let $I n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) \forall n \in N$. Determine $\bigcap_{n=1}^{\infty} I_n$ and is it Nhd. of each of its points?
6. Determine the limit points of the set $x = \left\{\frac{n}{n+1} \middle| n \in N\right\}$?
7. Determine all the limit point of set $x = \left\{\frac{1}{n} \middle| n \in N\right\}$
8. Find the limit points of the following sets :
   (i) $x = \left\{6 + \frac{1}{n} \middle| n \in N\right\}$
   (ii) $x = \left\{\frac{3n + 2}{2n + 1} \middle| n \in N\right\}$
9. Give an example of the following sets :
   (i) A set with only 0 limit point.
   (ii) A set whose derived set is empty.
   (iii) A set whose each points is limit points.

(iv) A set which is unbounded and having limit points.

(v) A set which is bounded having no limit points.

10. Find the derived set of the following sets :

(i) [1, 2] (ii) (1, 2)

(iii) [1, 2 [(iv)] 1, 2]

(v) ϕ (vi) $x = \{x\sqrt{2}\ |x \in Q|\}$

(vii) $x = \left\{ \frac{1 + (-1)^n}{n} \middle| n \in N \right\}$

(viii) $x = \{1 + 3^{-n} \mid n \in N\}$ (ix) $x = \left\{ 1 - \frac{4}{n} \middle| n \in N \right\}$

11. Define countable set with an example ?

12. Show that [0, 1] is not countable ?

13. Show that the set N × N is countable ?

14. Show that if x and y are two countable set then $x \cap y$ is also countable ?

15. Show that the set Q^+ of positive rational numbers is countable ?

ANSWERS EXERCISE 1(B)

1. (i) R – Q (ii) R

5. [0 , 1] , No 6. 1

7. 0

8. (i) 6 (ii) $\frac{3}{2}$

9. (i) $x = \left\{ \frac{1}{n} \ |n \in N \right\}$ (ii) N

(iii) R (iv) R

(v) Any finite set.

10. (i) [1, 2] (ii) [1, 2] (iii) [1, 2]

(iv) [1, 2] (v) ϕ (vi) R

(vii) {0} (viii) {1} (ix) {1}

***

Chapter 3

INFINITE SERIES

3.1. Infinite Series :

We consider the Sequence $< u_n >$ of real numbers then the expression of the form

$u_1 + u_2 + \ldots\ldots. + u_n + \ldots\ldots.$

is said to be an infinite series. We usually denote it by $\sum u_n$ or $\sum_{n=1}^{\infty} u_n$, u_n denote the n^{th} term of the series.

3.2. Series of positive terms :

If on infinite Series. $\sum u_n = u_1 + u_2 + \ldots\ldots. + u_n$ has all terms positive i.e. if $u_n > 0 \ \forall \ n$ then the series $\sum u_n$ is called series of positive terms.

3.3. Partial Sun :

Let $\sum u_n$ is an infinite series where terms may be positive or negative then, $S_n = u_1 + \ldots\ldots. + u_n$ is called n^{th} partial Sum. If $\sum u_n$, $S_1 = u_1$, u_1 first partial Sum, and $S_2 = u_1 + u_2$ = Second partial Sum.....

3.4. Nature of an Infinite Series :

An Infinite series $\sum u_n$ is (i) Convergence (or we can say convergent) it sequence $<S_n>$ of its partial Sum converges i.e.

If $\lim\limits_{n \to \infty} S_n$ = finite

(ii) Diverge (or we can say divergent) if sequence $<S_n>$ of its partial Sum diverges i.e. if

$\lim\limits_{n \to \infty} S_n = +\infty$ or $-\infty$

(iii) (a) Oscillates finitely if $<S_n>$ of its partial Sum Oscillates finitely i.e. If $<S_n>$ is bounded and neither converges nor diverges.

(b) Oscillates infinitely if $<S_n>$ of its partial Sum oscillates infinitely i.e. if $<S_n>$ is unbounded and neither converges nor diverges.

EXAMPLE

1. The Geometric series $1 + x + x^2 + \infty$ is

(i) Convergent if $-1 < x < 1$

(ii) Divergent if $x \geq 1$

(iii) Oscillates Finitely if $x = -1$

(iv) Oscillates infinitely if $x < -1$

Proof :

(i) When $-1 < x < 1$ i.e. $x \in]-1, 1[$

$$S_n = 1 + x + x^2 + ...+ \text{terms} = \frac{1.(1 - x^n)}{1 - x}$$

$$= \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$\lim_{n\to\infty} S_n = \lim_{x\to\infty} = \frac{1}{1-x} - \frac{x^n}{1-x}$$

$$= \frac{1}{1-x} = \text{Definite finite no.} |x| < 1$$

So, $x^n \to 0$ as $n \to \infty$

$\Rightarrow$ $< S_n >$ is convergent. So given series is convergent.

(ii) When $n \geq 1 : \to$

When $n = 1$,

Then $S_n = 1 + 1 + 1 + ...$ n terms $= n$

$$\lim S_n = \lim_{x\to\infty} n = \infty$$

So, $< S_n>$ divergent. When $x > 1$

$S_n = 1 + x + x^2$ ...... n terms

$$= 1 \cdot \frac{x^n - 1}{x - 1}$$

$$\lim S_n = \lim \frac{x^n - 1}{x - 1} = \lim \left[\frac{x^n}{x-1} - \frac{1}{x-1}\right]$$

$< S_n >$ is divergent. So given Series is divergent for $n > 1$

Hence given Geometric series is divergent for $x \geq 1$

(iii) $x = -1$

$S_n = 1 - 1 + 1 - 1 + \ldots\ldots n +$ terms

When $\quad n = \text{even}, \quad S_n = 0$

$\quad n = \text{odd}, \quad S_n = 1$

So, $< S_n >$ is bounded and neither converge nor diverges.

$< S_n >$ is oscillates finitely. So given series is oscillates finitely.

(iv) When $x < -1$

if $x < -1$, then $-x > 1$

So, $r = -x > 1$

$r^n \to \infty \quad$ as $n \to \infty$

$S_n = 1 + x + x^2 + \ldots\ldots n +$ terms

$$= \frac{1 - x^n}{1 - x} = \left(\frac{1 - (-r)^n}{1 + r}\right)$$

$$= \frac{1 - r^n}{1 + r} \text{ if n is even or } \frac{1 + r^n}{1 + r} \text{ if n is odd.}$$

$\lim S_n = \infty, -\infty$ according as n is odd and even.

$<S_n>$ is oscillates infinitely. So given Series oscillates infinitely.

2. Test the convergence or otherwise of the series 1 + 2 + 3....

Sol. :

$S_n \quad = 1 + 2 + 3 + \ldots\ldots. n$ terms

= Sum of first n natural nos

$$= \frac{n(n+1)}{2}$$

$$\lim S_n = \lim \frac{n(n+1)}{2} = \infty$$

Since $< S_n >$ is diverges to ∞, So given series is divergent.

3. Prove the series $1^2 + 2^2 + 3^2 + ...$ diverges to $+\infty$

Sol. :

$S_n = 1^2 + 2^2 + 3^2 + ...$ n terms

= Sum of sequare of first n natural no.

$$= \frac{n(n+1)(2n+1)}{6}$$

$$\lim S_n = \lim \frac{n(n+1)(2n+1)}{6} = \infty$$

$<S_n>$ diverges to $+\infty$

So given series diverges to $+\infty$

4. Prove that the series $\sum u_n$ where $u_n = -n$ diverges to $-\infty$

Sol. :

$S_n = -1, -2, -3$ ...... n terms

$$= \frac{-n(n+1)}{2}$$

$$\lim S_n = \lim \left[\frac{-n(n+1)}{2}\right]$$

$= -\infty$

$< S_n >$ diverges to $-\infty$ so given series diverges to $-\infty$

3. Test the convergence or other wise of 2 – 2 + 2 – 2.....

Sol. :

$S_n = 2 - 2 + 2 - 2 +$ .... n term

$$= \begin{cases} 0, & n \text{ is even} \\ 2, & n \text{ is odd} \end{cases}$$

$<S_n>$ is oscillates finite. So given series is oscillates finite.

Theorem 1: Necessary condition for convergence if an infinite series $\sum u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$

Proof :

Given $\sum u_n$ is convergent to prove lim $u_n = 0$

Since $\sum u_n$ is divergent $\Rightarrow$ $<S_n>$ is divergent when S_n denote n^{th} partial Sum of $\sum u_n$

$\Rightarrow$ lim S_n = finite and unique

= S (say)

$\Rightarrow$ lim $S_{n-1} = S$

$S_n - S_{n-1} = u_1 + u_2 + \ldots.. u_{n-1} + u_n -$

$(u_1 + u_2 + \ldots.. + u_{n-2} + u_{n-1})$

$= u_n$

taking lim as $n \to \infty$ both side we have

$$0 = \lim_{n\to\infty} u_n$$

$$\Rightarrow \lim_{n\to\infty} u_n = 0$$

Hence $\sum u_n$ is convergent. 0 then lim $u_n = 0$

Converse of the above theorem is not true we take the series--

$\sum u_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots \frac{1}{\sqrt{n}} + \ldots\ldots$ for this

Series, we have

$$u_n = \frac{1}{\sqrt{n}}$$

$$\lim u_n = \lim \frac{1}{\sqrt{n}} = 0$$

But this series is not convergent.

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} \rangle \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots. + \frac{1}{\sqrt{n}}$$

$$S_n = \frac{n}{\sqrt{n}} = \sqrt{n}$$

i.e. $S_n \rangle \sqrt{n}$ Which tends to infinity as $n \to \infty$

So, above series is divergent to $+\infty$

$\lim u_n = 0$

Note : If $\lim\limits_{n\to\infty} u_n \neq 0$, then $\sum u_n$ is not convergent.

Theorem 2: A series of Positive terms either converges or diverges to $+\infty$

Proof :

Let S_n be n^{th} partial Sum of positive term Series $\sum u_n$

$S_n = u_1 + u_2 + ...+ u_n$

$S_{n+1} = u_1 + u_2 + ... u_n + u_{n+1}$

$S_{n+1} - S_n = u_{n+1} > 0 \quad \forall\, n$

$\Rightarrow S_{n+1} > S_n \ \forall\, n$

$\Rightarrow < S_n >$ is monotonically increasing sequence.

Case I :

If $< S_n >$ is bounded. Then $< S_n >$ is convergent so $\sum u_n$ is convergent

Case II :

If $<S_n>$ is not bounded above then $<S_n>$ diverges to $+\infty$ i.e. $\sum u_n$ is diverges to $+\infty$

Corollary :

If $\sum u_n$ is a series of + ve terms and $\lim\limits_{n\to\infty} u_n \neq 0$ then the series diverges to $+\infty$

Theorem 3 : A positive term series $\sum u_n$ is converges if sequence $<S_n>$ of its partial Sum is bounded above.

Proof :

Let $<S_n>$ is bounded above. Since $\sum u_n$ is positive term series the sequence $<S_n>$ is monotonically increasing. We know a monotonically increasing bounded above sequence is converges so $<S_n>$ is converges. Hence $\sum u_n$ is converges.

Converse Part : Let $\sum u_n$ is converges. Then $<S_n>$ is also converges. Since Every convergent sequence is bounded so $<S_n>$ is bounded. Hence $<S_n>$ is bounded above.

Theorem 4 : Cauchy General principle of Convergence for series

An infinite series $\sum u_n$ converges iff to each $\in > 0$ $\exists$ a + ve integer m s.t. $\forall$ n > m we have $| u_{m+1} + u_{m+2} + ... + u_n | < \in$

Proof :

The series $\sum u_n$ is convergent $\Leftrightarrow$ $<S_n>$ sequence of its partial Sum is convergent

$\Leftrightarrow$ For each $\in > 0$ $\exists$ a + ve integer m s.t. $|S_n - S_m| < \in$ $\forall$ n > m

(By Cauchy general principle of convergence of sequence)

$\Leftrightarrow$ $|u_{m+1} + u_{m+2} + ...+ u_n| < \in$ $\forall$ n > m

Hence the result.

Theorem 5: Let m be any given positive integer then both the series $u_1 + u_2 + ... + u_{m+1} + ...$ and $u_{m+1} + u_{m+2} + ...$have same nature.

Proof :

let s_n and S_N be the n^{th} partial sum of given series

$u_1 + u_2 + ...u_{m+1}$ and $u_{m+1}, u_{m+2}..$

i.e. $S_n = u_1 + u_2 + ...+ u_n$

$S_N = u_{m+1} + u_{m+2} + ...+ u_{m+n}$

$S_N = u_1 + u_2 + ...+ u_{m+n} - (u_1 + u_2 + ...+ u_m)$

$S_N = S_{m+n} - S_m$

S_m is a fixed quantity due to Sum of finite no. of terms.

So $< S_n>$ and $< S_N>$ both have same nature i.e. both are together converge, diverge or oscillate.

Theorem 6: If $\sum u_n$ converges to u and $\sum v_n$ is converges to v then $\sum (u_n \pm v_n)$ converges to u + v respectively.

We have to prove. $\sum (u_n + v_n)$ converges to

$(u + v)$ Let $S_n = (u_1 + v_1) + (u_2 + v_2) + ...+ (u_n + v_n)$

then $S_n = (u_1 + u_2 + ...+ u_n) + (v_1 + v_2 + ...v_n)$

$= G_n + H_n$ Where $G_n = u_1 + u_2 + ...+ u_n$

$H_n = v_1 + v_2 + ...v_n$

$\lim S_n = \lim (G_n + H_n)$

$= \lim G_n + \lim H_n$
$= u + v$

$\because \sum u_n$ convergent u & $\sum v_n$ convergent to v

Hence $\sum(u_n + v_n)$ converges to $u + v$

Similarly we can prove $\sum(u_n - v_n)$ converges to $(u - v)$

Theorem 7:

(i) If $\sum u_n$ converges to u then $\sum k\, u_n$ converges to k u were k is a constant.

(ii) If $\sum u_n$ is divergent then $\sum k\, u_n$ is also divergent where $k \neq 0$.

Proof : (i)

Let $S_n = u_1 + ...+ u_n$, $G_n = k\, u_1 + ... + k\, u_n$
$G_n = k(u_1 + ...+ u_n) = k\, S_n$

$\lim G_n = \lim k\, S_n = k \lim S_n = k\, u$ $[\because \sum u_n$ converges to u]

$\Rightarrow \sum k\, u_n$ Converges to k u.

Proof : (ii)

Let $\sum u_n$ diverges to $+\infty$ then $\lim S_n = +\infty$

$\lim G_n = \lim (k\, u_1 + ... + k\, u_n)$
$= \lim k\,(u_1 + ...+ u_n)$
$= k \lim (u_1 + ...+ u_n)$
$= k \lim S_n$
$= k\,(+\infty)$
$= \infty$ or $-\infty$ it $k > 0$ or < 0

$\sum k\, u_n$ diverges $\forall\, k \neq 0$

Similarly if $\sum u_n$ diverges to $-\infty$, $\sum k\, u_n$ is diverges.

For $k \neq 0$

EXAMPLE

1. By using Cauchy's general principle of convergence show that the geris $\sum \frac{1}{n}$ does not converges ?

Sol. :

Let given series $\sum \frac{1}{n}$ is convergent. Take $\in = \frac{1}{2}$ by Cauchy general principle of convergence. $\exists$ a + ve integer m s.t.

$|u_{m+1} + ... + u_n| < \forall \ n > m$

i.e. $\left|\frac{1}{m+1} + ...+ \frac{1}{n}\right| < \frac{1}{2} \ \forall \ n \rangle m$

$\frac{1}{m+1} + ...+ \frac{1}{n} < \frac{1}{2} \quad \forall \ n \rangle m$

Now take n = 2 m.

$\frac{1}{m+1} + \frac{1}{m+2} ....+ \frac{1}{n} = \frac{1}{m+1} + \frac{1}{m+2} +........\frac{1}{m+m}$

$\Rightarrow \frac{1}{2m} + \frac{1}{2m} + ...+ \frac{1}{2m}$

$= \frac{m}{2m} = \frac{1}{2}$

Which is contradiction. So our assumption is wrong.

Hence the given series is not converges.

Theorem 8: P - series or the Auxiliary series :

The infinite series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + ...+ \frac{1}{n^p} + ...$ is converges if P > 1 and diverges if P ≤ 1

Proof : Case I

If P > 1

We write the given series as

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots \quad \text{---(1)}$$

$$\left(\frac{1}{1^p}\right) + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \ldots$$

$$= \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \ldots$$

$$= \frac{1}{1^P} + \frac{2}{(2)^P} + \left(\frac{2}{2^P}\right)^2 + \ldots \quad \text{---(2)}$$

The R.H.S. series (2) is geometric series with common ratio $\frac{1}{2^{p-1}} < 1$. So the R.H.S. series (2) is convergent. Hence the given series is also converges. if P = 1. then the series $\sum \frac{1}{n^p}$ is reduced in the form $\sum \frac{1}{n}$ i.e.

$$\sum \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots..$$

$$= \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \ldots\ldots..$$

$$\rangle \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \ldots\ldots..$$

$$= 1 + \frac{1}{2} + \frac{1}{2} \ldots\ldots..$$

The R.H.S. series is divergent as $\lim u_n = \frac{1}{2} \neq 0$

So, the given series is divergent.
If P < 1
We have P < 1 then $n^p < n$

i.e. $\frac{1}{n^p} \rangle \frac{1}{n}$

But $\sum \frac{1}{n}$ is divergent we prove above when P = 1

Hence $\sum \frac{1}{n^p}$ is divergent.

3.5. Comparison Test :

(i) Let $\sum u_n$ and $\sum v_n$ both series are of positive terms. if $\sum v_n$ convergent and $\exists$ a + ve constant k s.t. $u_n \leq k\, v_n \ \forall$ n then $\sum u_n$ is convergent.

(ii) If $\sum v_n$ is divergent and $\exists$ a + ve constant k s.t. $u_n \geq k\, v_n \ \forall$ n then $\sum u_n$ is divergent.

Proof :

Let $S_n = u_1 + u_2 + ...+ u_n$

$Gn = v_1 + v_2 + ... v_n$

$Sn \leq k\, v_1 + k\, v_2 + ... k\, v_n \qquad [\because u_n \leq k\, v_n \ \forall n]$

$= k\,(v_1 + v_2 + ...+ v_n)$

$= k\, Gn \ \forall n \qquad$ ---(1)

$\sum v_n$ is convergent So $< G_n >$ is convergent and Hence $< G_n >$ is bounded. So $\exists$ a + ve constant m s.t. $G_n < m \ \forall n$ ---(2)

So $Sn < k\, m \ \forall$ n From (1) & (2)

$\Rightarrow Sn < H$ Where $H = k\, m$

$\Rightarrow < Sn >$ is bounded above.

Also $\sum u_n$ is positive terms series, $< Sn >$ is monotonically increasing.

Since $< Sn >$ is monotonically increasing and bounded above so $< Sn >$ is divergent. Hence $\sum u_n$ is convergent. Similarly we can prove the second part i.e. it $\sum v_n$ is divergent, and $\exists$ positive constant k such that. $u_n \geq k\, v_n \ \forall$ n then $\sum u_n$ is divergent.

(ii) Let $\sum u_n$ and $\sum v_n$ both are series of positive terns. If $\sum v_n$ is convergent and $\exists$ a positive no. k s.t. $u_n \leq kv_n \ \forall n > m$ then $\sum u_n$ is converges if $\sum v_n$ is divergent and $\exists$ a positive no. k s.t. $u_n > k\, v_n \;\forall\; n > m$ then $\sum u_n$ is divergent.

Proof :

Let $Sn = u_1 + u_2 \ldots\ldots + u_n$

$u_n = v_1 + v_2 \ldots\ldots + v_n$

$u_n \le k\, v_n$ (given) $\forall\; n > m$

So, $u_{m+1} + u_{m+2} + \ldots\ldots + u_n \le k\, [v_{m+1} + \ldots\ldots + v_n]$

$\Rightarrow Sn - Sm \le k(Gn - Gm) \quad \forall\; n > m$

$\Rightarrow Sn \le k\, Gn + (Sm - k\, Gm) \quad \forall\; n > m$

$\Rightarrow Sn \le k\, Gn + H$ ---(1)

Where H = Sm – k Gn a fixed no.

Now, $\sum v_n$ is convergent. The sequence < Gn > is convergent and hence < Gn > is bounded above.

< Sn > is bounded above From (1)

Since $\sum u_n$ is + ve terms series. < Sn > is monotonically increasing.

Now, < Sn > is monotonically increasing and bounded above. So < Sn > is convergent.

Hence $\sum u_n$ is convergent.

For II nd Part.

Let $Sn = u_1 + u_2 + \ldots\ldots + u_n$ & $G_n = v_1 + v_2 \ldots\ldots + v_n$

$u_n > k\, v_n \quad \forall\; n > m$

So $u_{m+1} + \ldots\ldots + u_n > k\, (v_{m+1} + \ldots\ldots + v_n)$

$(Sn - Sm) > k\, (Gn - Gm) \quad \forall\; n > m$

$Sn > k\, Gn + Sm - k\, Gm \quad \forall\; n > m$

$Sn > k\, Gn + H$ Where H = Sm – k Gm a fixed no.

$\because \sum v_n$ is divergent, < Gn> is divergent so to each + ve no. h_1, $\exists$ a + ve integer m s.t. Gn $> h_1 \quad \forall\; n > m_1$

Let $m_2 = \max\, \{m, m_1\}$ then $Gn > h_1 \quad \forall\; n > m_2$

$Sn > k\, h_1 + H = k_1 \quad \forall\; n > m_2$

So < Sn> is divergent

Hence $\sum u_n$ is divergent.

(iii) If $\sum u_n$ & $\sum v_n$ both are series of positive terms and $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \ell$ where l is finite and non-zero then both series are converge or diverge together.

Proof :

Since $\sum u_n$ & $\sum v_n$ are series of positive terms

So, $\dfrac{u_n}{v_n} > 0$

$\therefore \lim \dfrac{u_n}{v_n} \geq 0$

$\Rightarrow l > 0$ [since l is non-zero and $\lim \dfrac{u_n}{v_n} = \ell$]

Now, $\lim \dfrac{u_n}{v_n} = \ell \Rightarrow$ to each $\in > 0\ \exists$ a + ve integer m s.t. $\left|\dfrac{u_n}{v_n} - \ell\right| \langle \in \quad \forall n \rangle m$

$\Rightarrow (l - \in)\, v_n < u_n < (l + \in)\, v_n \quad \forall n > m$

Take $\in > 0$ s.t. $(l - \in)\, v_n > 0$

From (1) $k_1 v_n < u_n < k_2 v_n \quad \forall n > m$ ---(1)

Where $k_1 = l - \in$

$k_2 = l + \in$

(a) Take $\sum u_n$ convergent.

$k_1 v_n < u_n \quad \forall n > m$

So, $\sum v_n$ is convergent [$\because \sum u_n$ is convergent.]

(b) Take $\sum u_n$ is divergent

$u_n < k_2 v_n$ From (1)

$\sum v_n$ is divergent [$\because \sum u_n$ is divergent.]

(c) Take $\sum v_n$ convergent.

$u_n < k_2 v_n \quad \forall n > m$ From (1)

$\sum u_n$ is convergent. [$\because \sum v_n$ is convergent.]

(d) Take $\sum v_n$ divergent.

$k_1 v_n < u_n$ From (1)

$\sum u_n$ divergent [$\because \sum v_n$ is convergent]

Hence both the series $\sum u_n$ and $\sum v_n$ convergent and divergent together.

Remark :

(i) $\sum u_n$ is converges if $\lim \frac{u_n}{v_n} = 0$ and $\sum v_n$ converges

(ii) $\sum u_n$ is diverges if $\lim \frac{u_n}{v_n} = \infty$ and $\sum v_n$ diverges

(I) If $\exists$ + ve integer m s.t. $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$ $\forall\, n \geq m$ where we have both the series $\sum u_n$ & $\sum v_n$ are positive terms series then if

(a) $\sum v_n$ is convergent then $\sum u_n$ is convergent.

(b) $\sum u_n$ is divergent then $\sum v_n$ is divergent.

Proof :

Let S_n and G_n are n^{th} Partial Sum of $\sum u_n$ and $\sum v_n$ respectively i.e.

$S_n = u_1 + u_2 + \ldots\ldots u_n$

$G_n = v_1 + v_2 \ldots\ldots + v_n$

Now, $\frac{u_m}{u_n} = \frac{u_m \cdot u_{m+1} \cdot u_{m+2} \ldots\ldots u_{n-1}}{u_{m+1} \cdot u_{m+2} \ldots\ldots u_{n-1}\, u_n}$

$= \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdot \ldots\ldots\ldots \cdot \frac{u_{n-1}}{u_n}$

$$\geq \frac{\nu_m}{\nu_{m+1}} \cdot \frac{\nu_{m+1}}{\nu_{m+2}} \cdot \,\ldots\ldots\ldots\, \cdot \frac{\nu_{n-1}}{\nu_n} \quad \forall\, n \geq m$$

$$= \frac{\nu_m}{\nu_n} \qquad \left[\because \frac{u_n}{u_{n+1}} \geq \frac{\nu_n}{\nu_{n+1}}\right] \quad \forall\, n \geq m$$

$$\Rightarrow \frac{u_m}{u_n} \geq \frac{\nu_m}{\nu_n}$$

$$\Rightarrow u_n \leq \frac{u_m}{v_m} v_m$$

$$\Rightarrow \; u_n \leq k\, \nu_m \quad \forall\, n \geq m$$

Where $k = \dfrac{u_m}{\nu_n}$

= Fixed no

Hence if $\sum \nu_n$ converges then $\sum u_n$ is converges and if $\sum \nu_n$ diverges then $\sum u_n$ diverges.

EXAMPLE

1. Show that the series

$\sum u_n = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots\ldots.. + \frac{1}{n^n} + \ldots\ldots$ **i.e.** $\sum u_n = \sum \frac{1}{n^n}$ **is convergent ?**

Sol. :

We have for all $n > 2 \quad n^n > 2^n$

$$\Rightarrow \frac{1}{n^n} \langle \frac{1}{2^n} \qquad \text{---(1)}$$

So take $u_n = \dfrac{1}{n^n}$ and $\nu_n = \dfrac{1}{2^n}$ ---(2)

From (1) & (2) $u_n < \nu_n \quad \forall\, n > 2$

So $\sum u_n = \sum \frac{1}{n^n}$ is convergent because $\sum \nu_n = \sum \frac{1}{2^n}$ is convergent as it is geometric series with common ratio ½.

2. Show that the series $\frac{1}{\log 2}+\frac{1}{\log 3}+\ldots\ldots\ldots\ldots+\frac{1}{\log n}+\ldots\ldots$ **divergent?**

Sol. :

We have for $n>1,\ \log n<n$

$$\Rightarrow \frac{1}{\log n}>\frac{1}{n} \quad \forall n>1$$

$$\Rightarrow \sum_{n=2}^{\infty}\frac{1}{\log n}>\sum_{n=2}^{\infty}\frac{1}{n}>$$

By P – test series $\sum_{n=2}^{\infty}\frac{1}{n}$ is divergent as P = 1. So given series $\frac{1}{\log 2}+\frac{1}{\log 3}+\ldots\ldots\ldots\ldots+\frac{1}{\log n}+\ldots\ldots$ is divergent.

3. Show that the series whose nth term is $\sin\frac{1}{n}$ **is divergent.**

Sol. :

$$u_n = \sin\frac{1}{n}$$

take $v_n = \frac{1}{n}$

$$\lim\frac{u_n}{v_n} = \lim\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \text{ finite and non-zero.}$$

So, by comparison test both the series $\sum u_n$ and $\sum v_n$ are convergent and divergent together. The series $\sum v_n = \sum\frac{1}{n}$ is diverges because $\sum\frac{1}{n^p}$ is divergent if $P \le 1$. Here $P = 1$

Hence the given series is divergent.

4. Discuss the convergence or divergence of the series whose n^{th} terms are

(i) $\dfrac{\sqrt{n}}{n^2+1}$ (ii) $\dfrac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$

(iii) $\dfrac{1}{1+\frac{1}{n}}$ (iv) $\tan^{-1}\dfrac{1}{n}$

Sol. :

(i) $u_n = \dfrac{\sqrt{n}}{n^2+1} = \dfrac{1}{n^{3/2}\left(1 + \dfrac{1}{n^2}\right)}$

take $\nu_n = \dfrac{1}{n^{3/2}}$

$\lim \dfrac{u_n}{\nu_n} = \lim \dfrac{1}{1+\dfrac{1}{n^2}} = 1$, finite and non-zero

So, by comparision tast both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the P – Series $\sum \dfrac{1}{n^{3/2}}$ is convergent as $P = \dfrac{3}{2} > 1$. So $\sum u_n$ is convergent.

(ii) $u_n = \dfrac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$

$$u_n = \frac{n^{\frac{2}{3}}\left(2 - \dfrac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{3}{4}}\left(3+\dfrac{2}{n^2}+\dfrac{5}{n^3}\right)^{\frac{1}{4}}}$$

$$= \frac{1}{n^{\frac{1}{12}}}\left[\frac{\left(2 - \frac{1}{n^2}\right)^{\frac{1}{3}}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}\right]$$

Take $v_n = \frac{1}{n^{\frac{1}{12}}}$, So, $\lim\frac{u_n}{v_n} = \lim \frac{\left(2 - \frac{1}{n^2}\right)^{\frac{1}{3}}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$

$= \frac{2^{\frac{1}{3}}}{3^{\frac{1}{4}}}$ Finite and non-zero.

So by comparison test both the series $\sum u_n$ and $\sum v_n$ are convergent or divergent together. Since the P – series $\sum\frac{1}{(n)^{\frac{1}{12}}}$ is divergent as $P = \frac{1}{12} \langle 1$ so the series $\sum u_n$ is convergent.

(iii) $u_n = \frac{1}{1+\frac{1}{n}}$

$\Rightarrow \lim u_n = \lim\frac{1}{1+\frac{1}{n}} = 1 \rangle 0$

So, the series $\sum u_n$ is divergent.

(iv) $u_n = \tan^{-1}\frac{1}{n}$

$u_n = \frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5}$ ....

$$u_n = \frac{1}{n}\left[1 - \frac{1}{3n^2} + \frac{1}{5n^4} \,.......\right]$$

Take $\nu_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{\nu_n} = \lim_{n\to\infty}\left[1 - \frac{1}{3n^2} + \frac{1}{5n^4} \,.......\right]$$

= 1 , Finite and non-zero.

So by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \frac{1}{n}$ is divergent as P = 1 So the given series $\sum u_n$ is divergent.

5. Test the convergence of the Following series

(i) $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \,.......$

(ii) $1 + \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \dfrac{4^4}{5^5} + \,.......$

Sol. :

(i) $\sum u_n = \dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5} + \,.......$

$$u_n = \frac{n}{n+1}$$

Now Proceed as Ex- 4(iii). The given series is divergent.

(ii) $1 + \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \dfrac{4^4}{5^5} + \,.......$

leaving the first term we have

$$\sum u_n = \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \,.......$$

$$u_n = \frac{n^n}{(n+1)^{n+1}}$$

Take $v_n = \frac{n^n}{n^{n+1}} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{n^n}{(n+1)^{n+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^n . n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \left\{ \frac{n^n}{(n+1)^n} \cdot \frac{n}{n+1} \right\}$$

$= \frac{1}{e}$ Finite and non-zero.

So by comparison test both the series $\sum u_n$ and $\sum v_n$ are convergent or divergent together. Since the P – Series $\sum v_n = \sum \frac{1}{n}$ is divergent as P = 1. So the given series $\sum u_n$ is also divergent.

6. Test the following series for convergence

(i) $\sum\left[(n^2+1)^{\frac{1}{2}} - n\right]$ (ii) $\sum\left[(n^3+1)^{\frac{1}{2}} - n^{\frac{3}{2}}\right]$

(iii) $\sum\left[(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}}\right]$ (iv) $\sum\left[(n^4+1)^{\frac{1}{2}} - (n^4-1)^{\frac{1}{2}}\right]$

Sol. :

(i) $\sum u_n = \left[(n^2+1)^{\frac{1}{2}} - 1\right]$

$u_n = (n^2+1)^{\frac{1}{2}} - n$

$$= \frac{\left(\sqrt{n^2+1} - n\right)\left(\sqrt{n^2+1} + n\right)}{\left(\sqrt{n^2+1} + n\right)}$$

$$= \frac{n^2+1 - n^2}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{n^2+1} + n} = \frac{1}{n}\left[\frac{1}{\sqrt{1+\frac{1}{n}} + 1}\right]$$

Take $\nu_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{\sqrt{1+\frac{1}{n}} + 1}\right]$$

$= \frac{1}{2}$ Finite and non-zero.

So, by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \nu_n = \sum \frac{1}{n}$ is divergent as P = 1 so the given series $\sum u_n$ is also divergent.

(ii) $\sum u_n = \sum\left[\left(n^3+1\right)^{\frac{1}{2}} - n^{\frac{3}{2}}\right]$

$$u_n = \left(n^3+1\right)^{\frac{1}{2}} - n^{\frac{3}{2}}$$

$$= \frac{\left(\left(n^3+1\right)^{\frac{1}{2}} - n^{\frac{3}{2}}\right)\left(\left(n^3+1\right)^{\frac{1}{2}} + n^{\frac{3}{2}}\right)}{\left(n^3+1\right)^{\frac{1}{2}} + n^{\frac{3}{2}}}$$

$$u_n = \frac{n^3 + 1 - n^3}{(n^3+1)^{\frac{1}{2}} + n^{\frac{3}{2}}} = \frac{1}{(n^3+1)^{\frac{1}{2}} + n^{\frac{3}{2}}}$$

$$u_n = \frac{1}{(n)^{\frac{3}{2}}} \left[\frac{1}{\left(1 + \frac{1}{n^3}\right)^{\frac{1}{2}} + 1} \right]$$

Take $\nu_n = \dfrac{1}{n^{3/2}}$

$$\frac{u_n}{\nu_n} = \frac{1}{\left(1 + \frac{1}{n^3}\right)^{\frac{1}{2}} + 1}$$

$$\lim \frac{u_n}{\nu_n} = \lim \frac{1}{\left(1 + \frac{1}{n^3}\right)^{\frac{1}{2}} + 1} = \frac{1}{2}, \text{ Finite and non-zero}$$

So, by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \nu_n = \sum \frac{1}{n^{3/2}}$ is convergent as $P = \frac{3}{2} \rangle 1$ So the series $\sum u_n$ is convergent.

(iii) $\sum \left[(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}} \right]$

$$u_n = \left[(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}} \right]$$

$$u_n = \frac{\left[(n+1)^{\frac{1}{2}} - n^{\frac{1}{2}}\right]\left[(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\right]}{\left[(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}\right]}$$

$$u_n = \frac{n+1-n}{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}} = \frac{1}{(n+1)^{\frac{1}{2}} + n^{\frac{1}{2}}} = \frac{1}{(n)^{1/2}}\left[\frac{1}{\left(1+\frac{1}{n}\right)^{\frac{1}{2}} + 1}\right]$$

Take $\dfrac{u_n}{\nu_n} = \dfrac{1}{(n)^{1/2}}$

$$\lim_{n\to\infty} \frac{u_n}{\nu_n} = \lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{n}\right)^{\frac{1}{2}} + 1}\right] = \frac{1}{2} \text{ Finite and non-zero.}$$

So by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \nu_n = \sum \frac{1}{n^{1/2}}$ is divergent as $P = \frac{1}{2} \langle 1$ So the given series $\sum u_n$ is divergent.

(iv) $\left|\dfrac{u_n+1}{u_n} - \ell\right| < \in \quad \forall\, n \geq m$

$$u_n = (n^4+1)^{\frac{1}{2}} - (n^4-1)^{\frac{1}{2}}$$

$$u_n = \frac{\left[(n^4+1)^{\frac{1}{2}} - (n^4-1)^{\frac{1}{2}}\right]\left[(n^4+1)^{\frac{1}{2}} + (n^4-1)^{\frac{1}{2}}\right]}{\left[(n^4+1)^{\frac{1}{2}} + (n^4-1)^{\frac{1}{2}}\right]}$$

$$u_n = \frac{(n^4+1)-(n^4-1)}{(n^4+1)^{\frac{1}{2}} + (n^4-1)^{\frac{1}{2}}} = \frac{2}{(n^4+1)^{\frac{1}{2}} + (n^4-1)^{\frac{1}{2}}}$$

$$u_n = \frac{2}{n^2\left[\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}} + \left(1-\frac{1}{n^4}\right)^{\frac{1}{2}}\right]}$$

Take $\nu_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{u_n}{\nu_n} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n^4}\right)^{\frac{1}{2}} + \left(1-\frac{1}{n^4}\right)^{\frac{1}{2}}} = \frac{2}{2} = 1 \text{ Finite and non-zero.}$$

So, by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \nu_n = \sum \frac{1}{n^2}$ is convergent as P = 2 > 1 so the given series $\sum u_n$ is also convergent.

7. Show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$ is convergent ?

Sol. :

$$u_n = \frac{1}{2^n + 3^n} = \frac{1}{3^n}\left[\frac{1}{1+\left(\frac{2}{3}\right)^n}\right]$$

$$\lim_{n\to\infty} \frac{u_n}{\nu_n} = \lim_{n\to\infty}\left(\frac{1}{1+\left(\frac{2}{3}\right)^n}\right) = 1 \text{ Finite and non-zero.}$$

So by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \nu_n = \sum \frac{1}{3^n}$ is convergent because $\sum \frac{1}{3^n}$ is geometric series with common ratio $\frac{1}{3} < 1$. Hence the given series $\sum u_n$ is convergent.

8. Test the convergence for the series

(i) $\sum \frac{1}{n\left(a + \frac{b}{n}\right)}$ (ii) $\sum \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$

Sol. :

(i) $\sum u_n = \sum \frac{1}{n\left(a + \frac{b}{n}\right)}$

$$u_n = \frac{1}{n\left(a + \frac{b}{n}\right)} = \frac{1}{n^a \cdot n^{b/n}}$$

Take $\nu_n = \frac{1}{n^a}$

$$\lim_{n\to\infty}\frac{u_n}{\nu_n} = \lim_{n\to\infty}\frac{1}{n^{b/n}} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^b = 1 \quad \left(\text{as } \lim_{n\to\infty} n^n = 1\right)$$

Finite and non-zero.

By comparison test both the series in convergent or divergent together. Since the series $\sum \nu_n = \sum \frac{1}{n}$ is convergent if a > 1 and divergent a ≤ 1

So the given series $\sum u_n$ is convergent if a > 1 and divergent if a ≤ 1

(ii) $\sum u_n = \sum \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$

$$u_n = \frac{1}{n^3}\left[\left(\frac{n+2}{n+3}\right)\right]^n$$

Choose $\nu_n = \frac{1}{n^3}$, $\lim_{n\to\infty}\frac{u_n}{\nu_n} = \lim_{n\to\infty}\left(\frac{n+2}{n+3}\right)^n$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}$$

$$= \frac{e^2}{e^3} = \frac{1}{e} \text{ Finite and non-zero.}$$

So by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent ot divergent together. since the series $\sum \nu_n = \sum \frac{1}{n^3}$ is convergent as P = 3 > 1, So the given series $\sum u_n$ is convergent.

3.6. Cauchy's root test :

Let $\sum u_n$ infinite series of positive terms and $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \ell$

If (a) $l < 1$ then the series is convergent.

(b) $l > 1$ then the series is divergent.

(c) $l = 1$ then the test fails.

Proof :

$\lim\limits_{n\to\infty} (u_n)^{\frac{1}{n}} = \ell$ or $u_n > 0 \quad \forall$ n and $(u_n)^{\frac{1}{n}}$ presents for nth. root of u_n. $\lim\limits_{n\to 0} (u_n)^{\frac{1}{n}} = \ell$, to each $\in > 0\ \exists$ a + ve integer m such then.

$$\left|(u_n)^{\frac{1}{n}} - \ell\right| \langle \in \quad \forall n \rangle m$$

$$\ell - \in \langle u_n^{\frac{1}{n}} \langle \ell + \in \quad \forall n \rangle m$$

$(l - \in)^n < u_n < (l + \in)^n \quad \forall\ n > m$ ---(1)

(a) If $l < 1$ then take $\in > 0$ s.t. $s = l + \in < 1$

So $0 \leq l < l + \in = S < 1$

From, (i) $u_n < (l + \in)^n = S_n \quad \forall\ n > m$

By Comparison test $\sum u_n$ is convergent as $\sum S_n$ is

Convergent because $\sum S_n$ is geometric series with common ratio $s < 1$.

(b) If $l > 1$, then take $\in > 0$ s.t. $S = l - \in > |$

(1) gives $(l - \in)^n < u_n \quad \forall\ n > m$

$= S_n < u_n \quad \forall\ n > m$

By comparison test $\sum u_n$ divergent as $\sum S_n$ is divergent series because $\sum S_n$ is geometric series with common ratio $s > 1$.

(c) For $l = 1$

Consider the series $\sum \frac{1}{n^2}$ this series is divergent as $P = 2 > 1$,

$$u_n = \frac{1}{n^2} \Rightarrow (u_n)^{\frac{1}{n}} = \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \left(\frac{1}{n^{\frac{1}{n}}}\right)^2$$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2 = 1$$

This shows for $l = 1$ a series may be convergent.

We consider the another series $\sum \frac{1}{n}$

$\sum \frac{1}{n}$ is divergent as P = 1

$$u_n = \frac{1}{n} \Rightarrow (u_n)^{1/n} = \frac{1}{(n)^{1/n}}$$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{(n)^{1/n}} = 1$$

This show for $l = 1$, series may be divergent.

Hence for $l = 1$, Cauchy root test fail.

1. Show by Cauchy root test the series $\sum \frac{1}{(\log n)^n}$ is convergent.

Sol. :

$$\sum u_n = \sum \frac{1}{(\log n)^n}$$

$$u_n = \frac{1}{(\log n)^n}$$

$$(u_n)^{1/n} = \left[\frac{1}{(\log n)^n}\right]^{1/n} = \frac{1}{\log n}$$

Now, taking limit as $n \to \infty$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$$

Hence by Cauchy root test given series is convergent.

2. Test the convergence of the following series by Cauchy root test

(i) $\sum \frac{x^n}{\angle n}$ (ii) $\sum \left(1+\frac{1}{n}\right)^{-n^2}$

(iii) $\sum\left(\frac{n+1}{n+2}\right)^2 \cdot x^n, \; x \rangle 0$

Sol. :

(i) $\sum \frac{x^n}{\angle n} \Rightarrow u_n = \frac{x^n}{\angle n}$

$$(u_n)^{1/n} = \left(\frac{x^n}{\angle n}\right)^{1/n} = \frac{x}{(\angle n)^{1/n}}$$

Taking limit as $n \to \infty$ we have

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{x}{(\angle n)^{1/n}} = \lim_{n\to\infty} \left|\frac{x}{(\angle n)^{1/n}} \cdot \frac{n}{n}\right|$$

$$= \lim_{n\to\infty}\left[\left(\frac{n^n}{\angle n}\right)^{1/n} \cdot \frac{x}{n}\right] = \lim_{n\to\infty}\left(\frac{n^n}{\angle n}\right)^{1/n} \cdot \lim_{n\to\infty}\frac{x}{n}$$

$= e \,.\, 0 = 0 < 1$

So by Cauchy root test given series is convergent.

(ii) $\sum\left(1+\frac{1}{n}\right)^{-n^2}$

$$u_n = \left(1 + \frac{1}{n}\right)^{-n^2} \Rightarrow \lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \langle 1$$

So by Cauchy root test given series is convergent.

(iii) $\sum\left(\frac{n+1}{n+2}\right)^2 \cdot x^n, \; x \rangle 0$

$$u_n = \left(\frac{n+1}{n+2}\right)^n \cdot x^n$$

$$\Rightarrow \lim_{x\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)^n \cdot x^n\right]^{1/n} = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)\cdot x$$

Therefore by Cauchy root test if $x < 1$, then the given series is convergent if $x > 1$, then the given series is divergent if $x = 1$ then the test is fail.

So take $x = 1$

W have $u_n = \left(\frac{n+1}{n+2}\right)^n$

$$\lim u_n = \lim \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{1}{e} \rangle 0$$

So the series is divergent when $x = 1$. Hence the given series is convergent when $x < 1$ and divergent when $x \geq 1$

3. Test the convergence of the following series.

(i) $\frac{1^3}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \ldots\ldots \frac{x^3}{3^n} + \ldots\ldots$

(ii) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots\ldots$

Sol. :

(i) $\sum u_n = \sum \frac{x^3}{3^n} \Rightarrow u_n = \frac{x^3}{3^3}$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{x^3}{3^3}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{x^{\frac{1}{n}}}{3}\right)^3 = \frac{1}{3} < 1$$

By Cauchy root test given series is convergent.

(ii) $\Sigma u_n = \Sigma\left[\frac{(x+1)^{n+1}}{x^{n+1}} - \frac{x+1}{x}\right]^{-n}$

$$\Rightarrow u_n = \left[\left(\frac{(x+1)}{x}\right)^{n+1} - \frac{x+1}{x}\right]^{-n}$$

$$\Rightarrow \lim_{x\to\infty}(u_n)^{\frac{1}{n}} = \lim_{x\to\infty}\left[\left(\frac{(x+1)}{x}\right)^{n+1} - \frac{x+1}{x}\right]^{-1}$$

$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{n+1} - \left(1+\frac{1}{x}\right)\right]^{-1}$$

$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{-1}\left\{\left(1 + \frac{1}{x}\right)^{n} - 1\right\}^{-1}\right]$$

$$= \frac{1}{e-1} \langle 1$$

By Cauchy root test given series is convergent.

3.7. D'Alembert's Ratio test :

If $\sum u_n$ is a series of positive terms and

$$\lim_{n\to\infty}\frac{u_n+1}{u_n} = \ell$$

(i) If $l < 1$ then $\sum u_n$ is converge

(ii) If $l > 1$ then $\sum u_n$ is diverges

(iii) If $l = 1$ then test fail

We state the D'Alembert's Ratio test in other way as---

If $\sum u_n$ is a series of positive terms and $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \ell$

(i) If $l > 1$ then $\sum u_n$ is converge

(ii) If $l < 1$ then $\sum u_n$ is diverges

(iii) If $l = 1$ then test fail

Proof :

It is given that $\sum u_n$ is positive terms series so $\frac{u_n+1}{u_n} > 0$

$$= \frac{u_n+1}{u_n} \rangle\ 0 \Rightarrow \lim \frac{u_n+1}{u_n} = \ell \geq 0$$

Since $\lim \frac{u_n+1}{u_n} = \ell \Rightarrow$ to each $\in \rangle\ 0$

$\exists$ a Integer 'm' such then

$$\left|\frac{u_n+1}{u_n} - \ell\right| \langle\ \in \quad \forall\, n \geq m$$

$$(\ell - \in) \langle\ \frac{u_{n+1}}{u_n} \langle\ \ell + \in \quad \forall\, n \geq m \qquad \text{---(1)}$$

By Putting n = m, m + 1, ..., n – 1 in (1) we get (n – m) inequality we multiply the corresponding side of the inequality we get $(\ell - \in)^{n-m} \langle\ \frac{u_n}{u_m} \langle\ (\ell + \in)^{n-m}$ ---(2)

Consider $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_m} = \ell \langle\ 1$

We take $\in > 0$ s.t. $l + \in = S < 1$

So we have $0 \leq l < S < 1$

Now (2) gives $\frac{u_n}{u_m} \langle\ S^{n-m}$

i.e. $u_n < u_m S^{n-m} \quad \forall\, n \geq m$

or. $u_n \langle\ \frac{u_m}{S^m} \cdot S^n$

or $u_n < k\,S^n$ Where $k = \dfrac{u_m}{S^m}$

$\sum S^n$ is convergent series because $\sum S^n$ is a geometric series whose common ratio is s < 1. By comparison test series $\sum u_n$ is convergent.

Now, Consider $\lim \dfrac{u_{n+1}}{u_n} = \ell \rangle 1$

We take $\in > 0$ S.t. $l - \in = S > 1$

we have $(\ell - \in)^{n-m} \langle \dfrac{u_n}{u_m}$ From ---(2)

i.e. $S^{n-m} \langle \dfrac{u_n}{u_m}$

i.e. $\dfrac{S^n}{S^m} \langle \dfrac{u_n}{u_m} \Rightarrow S^n \dfrac{u_m}{S_m} \langle u_n$

$\Rightarrow S^n k_1 < u_n$

Where $k_1 = \dfrac{u_m}{S^m}$

$\sum S^n$ is divergent series because $\sum S^n$ is a geometric series whose common ratio S > 1. By comparison test the series $\sum u_n$ is divergent.

Now we take two series for the case

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$$

Consider $\sum \dfrac{1}{n^2}$ this series is convergent as P = 2 > 1

$u_n = \dfrac{1}{n^2}$, $u_{n+1} = \dfrac{1}{(n+1)^2}$ · So $\lim \dfrac{u_{n+1}}{u_n} = \lim \dfrac{n^2}{(n+1)^2}$

$= 1 = l$

Consider $\sum \frac{1}{n}$. This series is divergent as P = 1

$$u_n = \frac{1}{n}, \quad u_{n+1} = \frac{1}{n+1}, \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1}$$

$= 1 = l$

From above we conclude that $l = 1$. D'Alembert ratio test cannot decide the behavior of series.

Hence if $l = 1$ test fail.

Remark 1 :

It $\sum u_n$ is a series of positive terms and $\lim \frac{u_{n+1}}{u_n} = \infty$ then the series $\sum u_n$ is divergent. If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty$ then the series is convergent.

1. By D'Alembert's ratio test show that the following series are convergent.

(i) $1 + \frac{3}{\angle 2} + \frac{5}{\angle 3} + \frac{7}{\angle 4} + \ldots\ldots.$

(ii) $1 + \frac{2^P}{\angle 2} + \frac{3^P}{\angle 3} + \frac{4^P}{\angle 4} + \ldots\ldots.$

(iii) $\frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \ldots\ldots.$

Sol. :

(i) $\sum u_n = 1 + \frac{3}{\angle 2} + \frac{5}{\angle 3} + \frac{7}{\angle 4} + \ldots\ldots. \frac{2n-1}{\angle n} + \ldots..$

$$u_n = \frac{2n-1}{\angle n}$$

$$u_{n+1} = \frac{2n+1}{\angle 2n+1}$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{2n+1}{\angle 2n+1} \cdot \frac{\angle n}{(2n-1)}$$

$$= \lim \frac{2n+1}{2n-1} \cdot \frac{1}{n+1} = 0 \langle 1$$

Hence by D'Alembert's ratio test given series is convergent.

(ii) $\sum u_n = 1 + \frac{2^P}{\angle 2} + \frac{3^P}{\angle 3} + \frac{4^P}{\angle 4} + \ldots\ldots.$

$$u_n = \frac{n^P}{\angle n} \quad , \; u_{n+1} = \frac{(n+1)^p}{\angle n+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{\angle n+1} \cdot \frac{\angle n}{n^P} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^P \cdot \frac{1}{n+1}$$

$= 0 < 1$

Hence by D'Alembert's ratio test given series is convergent.

(iii) $\sum u_n = \frac{1}{1+2} + \frac{2}{1+2^2} + \frac{3}{1+2^3} + \ldots\ldots + \frac{n}{1+2^n} + \ldots\ldots$

$$u_n = \frac{n}{1+2^n} \quad , \; u_{n+1} = \frac{n+1}{1+2^{n+1}}$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \left[\frac{n+1}{1+2^{n+1}} \cdot \frac{1+2^n}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[\left(1+\frac{1}{n}\right)\frac{\left(1+\frac{1}{2^n}\right)}{1+\frac{1}{2^{n+1}}}\right] = \frac{1}{2} \langle 1$$

Hence by D'Alembert's ratio test given series is convergent.

2. Show that the series $\sum \frac{\angle n}{n^n}$ is convergent ?

Sol. :

$$u_n = \frac{\angle n}{n^n} \quad , \quad u_{n+1} = \frac{\angle n + 1}{(n + 1)^{n+1}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim \left[\frac{\angle n+1}{(n+1)^{n+1}} \cdot \frac{n^n}{\angle n} \right]$$

$$= \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \langle 1$$

Hence by 'DAlembert's ratio test given series is convergent.

3. Test for convergence or divergence of the series

(a) $1 + 2x + 3x^2 + 4x^3$........

(b) $a + (a+d)\, x + (a + 2d)\, x^2 +$ ....

Sol. :

$$\sum u_n = 1 + 2x + 3x^2 + 4x^3 + ...$$

$u_n = n\, x^{n-1}$

$u_{n+1} = (n + 1)\, x^n$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim \left(\frac{n + 1}{n} \right) x = x$$

So the given series is convergent if $x < 1$ and divergent if $x > 1$.

If $x = 1$ then $\sum u_n = \sum n = \sum \frac{1}{n^{-1}}$ which is divergent as $P = -1 < 1$

(b) $\sum u_n = a + (a + d)\, x + (a + 2d)\, x^2 + ... + [a + (n-1)d]\, x^{n-1} + ...$ $\qquad u_n = [a + (n - 1)\, d\,]$ x^{n-1}

$u_{n+1} = (a + nd)\, x^n$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim \left(\frac{a + nd}{a + (n - 1)\, d} \cdot x \right)$$

$$= \lim \left\{ \frac{n}{n-1} \cdot \frac{\left[\frac{a}{n} + d\right]}{\left[\frac{a}{n-1} + d\right]} \cdot x \right\}$$

$= x$

By D'Alembert's ratio test given series is convegent if $x < 1$ and divergent if $x > 1$. if $x = 1$ then

$u_n = a + (n - 1)\, d$

we have $\lim u_n = \infty$

and $\sum u_n$ is + ve terms series. So it is divergent.

4. Test the series for convergence

$$1 + \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \frac{x^3}{\angle 3} + ...\infty$$

leaving first terms we have the series

$$\frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \frac{x^3}{\angle 3} + ...$$

$$u_n = \frac{x^2}{\angle 2}, u_{n+1} = \frac{x^{n+1}}{\angle x+1}$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{x^{n+1}}{\angle x+1} \cdot \frac{\angle n}{x^n} = \lim \frac{1}{n+1} \cdot x = 0 \langle 1$$

Hence by D'Alembert's ratio test given series is convergent.

5. Test for convergence of the following series

(i) $\sum \frac{a^n}{a^n + x^n}$ (ii) $\frac{1}{x^n + x^{-n}}$

Sol. :

$$\sum u_n = \sum \frac{a^n}{a^n + x^n}$$

$$u_n = \frac{a^n}{a^n + x^n} \cdot u_{n+1} = \frac{a^{n+1}}{a^{n+1} + x^{n+1}}$$

$$\lim \frac{u_{n+1}}{u_n} = \lim\left[\frac{a^{n+1}}{a^{n+1}+x^{n+1}} \cdot \frac{a^n + x^n}{a^n}\right]$$

$$= \lim \frac{a}{x}\left[\frac{1+\left(\frac{a}{x}\right)^n}{1+\left(\frac{a}{x}\right)^{n+1}}\right]$$

If x > a then $\lim \frac{u_n+1}{u_n} < 1$ i.e. given series is convergent. If x < a then

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{a \cdot a^n\left[1+\left(\frac{x}{a}\right)^n\right]}{a^{n+1}\left[1+\left(\frac{x}{a}\right)^{n+1}\right]} = 1 \text{ test fail.}$$

For this case,

$$\lim u_n = \lim \frac{a^n}{a^n + x^n} = 1 \langle 0$$

So the given series divergent for x < a

If x = a then the series is reduced in the form $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots.$ is divergent.

Hence the given series is convergent if x > a and divergent if $x \leq a$

(ii) $\sum u_n = \sum \frac{1}{x^n + x^{-n}}$

$$u_n = \frac{1}{x^n + x^{-n}} \quad , \quad u_{n+1} = \frac{1}{x^{n+1} + x^{-n+1}}$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{x^n + x^{-n}}{x^{n+1} + x^{-(n+1)}} = \lim\left[\frac{x^{2n}+1}{x^{2n}+1} x\right]$$

Now there arise three cases

(i) if x < 1

$$\lim \frac{u_{n+1}}{u_n} = \lim \left[\frac{x^{2n}+1}{x^{2n}+1} x \right] = x$$

By D'Alembert's test given series is convergent.

(ii) if x > 1

$$\lim \frac{u_{n+1}}{u_n} = \lim \left[\frac{x^{2n}+1}{x^{2n}+1} x \right] = x$$

$$= \lim \left[\frac{1+\frac{1}{x^{2n}}}{1+\frac{1}{x^{2n}}} \frac{1}{x} \right] = \frac{1}{x} < 1$$

So the given series is convergent.

(iii) If x = 1

Then $\sum u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots$

$S_n = n^{th}$ partial Sum $= \frac{n}{2}$

$$\lim S_n = \lim \frac{n}{2}$$

So the series is divergent.

Hence the given series is convergent if x > 1 or x < 1 and divergent x = 1

6. Test the following series for convergence

(i) $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n, \quad (x \rangle 0)$

(ii) $\sum \frac{n}{n^2+1} \cdot x^n, \quad x \rangle 0$

Sol. :

(i) $\sum u_n = \sum \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n$

$$u_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} \cdot x^n \ , \ u_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} \cdot x^{n+1}$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{\sqrt{n+1}}{\sqrt{n}} \ \frac{\sqrt{n^2+1}}{\sqrt{(n+1)^2+1}} \cdot x$$

$$\lim \left\{ \left[\sqrt{\left(1+\frac{1}{n}\right)} \ \sqrt{\frac{n^2+1}{n^2+2+2n}} \cdot x \right] \right\}$$

$$\lim \left[\sqrt{\left(1+\frac{1}{n}\right)} \ \sqrt{\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}} \cdot x \ \right] = x$$

If $x < 1$ then by ratio test given series is convergent.

If $x > 1$ then by ratio test given series is divergent.

If $x = 1$ then the ratio test fail.

Now, take $x = 1$ the given series is reduced

$$\sum u_n = \sum \sqrt{\frac{n}{n^2+1}}$$

$$u_n = \sqrt{\frac{n}{n^2+1}} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1 + \frac{1}{n}}}$$

Take $\quad \nu_n = \dfrac{1}{\sqrt{n}}$

$$\lim \frac{u_n}{\nu_n} = \lim \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$

Finite and non-zero. So by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \nu_n = \sum \dfrac{1}{n^{1/2}}$

is divergent as $P = \frac{1}{2} < 1$ So the series $\sum u_n$ is divergent.

Hence the given series is convergent if x < 1 and divergent if x ≥ 1.

(ii) $\sum u_n = \sum \frac{n}{n^2+1} \cdot x^n, \quad x > 0$

$$u_n = \sum \frac{n}{n^2+1} \cdot x^n, \; u_{n+1} = \frac{n+1}{(n+1)^2+1} \cdot x^{n+1}$$

$$\lim \frac{u_{n+1}}{u_n} = \lim \left[\frac{n+1}{(n+1)^2+1} \cdot \frac{n^2+1}{n} \cdot x \right]$$

$$= \lim \left[\frac{(n+1)(n^2+1)}{(n^2+2n+2)n} . x \right]$$

$$= \lim \left[\left(1+\frac{1}{n}\right) \cdot \frac{\left(1+\frac{1}{n^2}\right)}{\left(1+\frac{2}{n}+\frac{2}{n^2}\right)} \cdot x \right]$$

$= x$

If x < 1 then by ratio test given series is convergent.

If x > 1 then by ratio test given series is divergent.

If x = 1 then the ratio test fail.

When x = 1 then the series is $\sum u_n = \sum \frac{n}{n^2+1}$

$$u_n = \frac{n}{n^2+1} = \frac{1}{n} \cdot \left(\frac{1}{1+\frac{1}{n^2}} \right)$$

take $\quad \nu_n = \frac{1}{n}$

$\lim \frac{u_n}{\nu_n} = \lim \frac{1}{1+\frac{1}{n^2}} = 1$ Finite and non-zero so by comparison test both the series $\sum u_n$ and $\sum \nu_n$ are convergent or divergent together. Since the series $\sum \nu_n = \sum \frac{1}{n}$ is divergent because P = 1.

So the series $\sum u_n$ is divergent.

Hence the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

Improper Integral :

An integral which have the form $\int_{a \in R}^{\infty} f(x)dx$ is said to be improper integral

If $F(S) = \int_{a \in R}^{S} f(x)\,dx$, $S \in [a, \infty[$

and $F(S) \to$ finite limit 'l' as $S \to \infty$ then $\int_{a \in R}^{\infty} f(x)dx$ is converge to l other wise divergent.

3.8. Cauchy's Integral Test :

Theorem 9:

Consider f(x) is any non-negative monotonically decreasing integrable function on $[1, \infty[$ then the series $\sum_{n-1}^{\infty} f(x)$ and improper integral $\int_{1}^{\infty} f(x)\,dx$ converge and diverge together.

Proof :

It is given that f(x) is non-negative so this implies $f(x) \geq 0 \ \forall\, x \in [1, \infty[$. Therefore the given series $\sum f(x)$ has non-negative terms. We have $n \in N$ s.t. $n \leq x \leq n+1$ for any pt. $x \in [1, \infty[$.

So, $f(n) \geq f(x) \geq f(n+1)$ because ---(1)

f(x) is monotonically decreasing on $[1, \infty[$

Now integrate (1) From n to n + 1 we have

$$\int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

$$\Rightarrow f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1) \qquad \text{---(2)}$$

Put $n = 1$ in (2) we get

$$f(1) \geq \int_1^2 f(x)\,dx \geq f(2)$$

Put $n = 2$ in (2) we get

$$f(2) \geq \int_2^3 f(x)\,dx \geq f(3)$$

Put $n = 3$ in (2) we get

$$f(3) \geq \int_3^4 f(x)\,dx \geq f(4)$$

Put $n = n-1$ in (2) we get

$$f(n-1) \geq \int_{n-1}^{n} f(x)\,dx \geq f(n)$$

Add the above inequality we have

$$Sn - f(n) \geq \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots + \int_{n-1}^{n} f(x)\,dx \geq Sn - f(1)$$

Where $Sn = f(1) + \ldots\ldots + f(n) = n^{th}$ partial Sum of the series $\sum f(n)$.

$$\Rightarrow Sn - f(n) \geq \int_1^n f(x)\,dx \geq Sn - f(1)$$

$$\Rightarrow f(n) \leq Sn - \int_1^n f(x)\,dx \leq f(1) \qquad \text{---(3)}$$

Take $Sn - \int_1^n f(x)\,dx = h_n \ \forall\, n \in N$

Now we show that $< hn >$ is monotonically decreasing sequence.

$$h_{n+1} - h_n = S_{n+1} - \int_1^{n+1} f(x)\,dx - S_n + \int_1^n f(x)\,dx$$

$$= (S_{n+1} - S_n) - \left[\int_1^{n+1} f(x)\,dx - \int_1^n f(x)\,dx\right]$$

$$= f(n+1) - \left[\int_1^n f(x)\,dx - \int_n^{n+1} f(x)\,dx - \int_1^n f(x)\,dx\right]$$

$$= f(n+1) - \int_n^{n+1} f(x)dx \le 0$$

So $h_{n+1} \le h_n \quad \forall\, n \in N$

$\Rightarrow \langle h_n \rangle$ is monotonically decreasing.

From (3) $h_n \ge f(n) \ge 0$. So $\langle h_n \rangle$ is bounded below.

From above $\langle h_n \rangle$ is convergent.

$$h_n = S_n - \int_1^n f(x)\,dx \quad \Rightarrow \quad S_n = h_n + \int_1^n f(x)\,dx$$

and $\langle h_n \rangle$ is convergent. So sequence $\langle S_n \rangle$ and $\int_1^n f(x)\,dx$ converge or diverge together.

Consequently $\sum f(n)$ and the improper integral $\int_1^\infty f(x)\,dx$ both are converge or diverge together.

EXAMPLE

1. By using integral test, show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge if P > 1 and diverge if P ≤ 1?

Sol. :

$$f(n) = \frac{1}{n^p}$$

$f(x) = \frac{1}{x^p}$, f(x) is non-negative and decreasing on [1, ∞[

When P = 1

$$\text{Take In} = \int_1^n f(x)\,dx = \int_1^n \frac{1}{x}\,dx = \log^n$$

$$\text{So, } \int_1^\infty f(x)\,dx = \lim_{n\to\infty} \log^n = \infty$$

Therefore by Integral test both.

Integral and the series converge together. Since integral is divergent so the series $\sum \frac{1}{n^p}$ is divergent.

When $P \neq 1$,

$$\text{Then In} = \int_1^n \frac{1}{x^p}\,dx$$

$$= \left(\frac{x^{-p+1}}{-P+1}\right)_1^n = \frac{1}{1-P}\left(\frac{1}{x^{p-1}}\right)^n$$

$$= \frac{1}{1-P}\left(\frac{1}{x^{p-1}} - 1\right) = \frac{1}{P-1}\left(1 - \frac{1}{n^{P-1}}\right)$$

Now there are two sub cases

(a) When P > 1

$$\text{Then In} = \frac{1}{P-1}\left(1 - \frac{1}{n^{P-1}}\right) \qquad P - 1 \text{ is + ve}$$

$$\lim_{n\to\infty} \text{In} = \frac{1}{P-1} = \text{Finite}$$

So, $\int_1^\infty f(x)\,dx$ Converge. By integral test given series is also convergent.

(b) When $0 \leq P < 1$

$$I_n = \frac{1}{P-1}\left(1-\frac{1}{n^{p-1}}\right) = \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right) \quad P - 1 \text{ is negative}$$

$$\lim_{n\to\infty} I_n = \frac{1}{1-P}(+\infty) = \infty$$

So $\int_1^{\infty} f(x)\,dx$ is divergent.

By Integral test given series is divergent.

Hence the given series $\sum_{n=1}^{\infty} \frac{1}{n^P}$ convergent if $P > 1$ and divergent if $0 < P \leq 1$.

2. By use of Integral test discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^P}$, $P \rangle 0$ **?**

Sol. :

$$f(n) = \frac{1}{n(\log n)^P} \quad , \ P \rangle 0$$

$$\Rightarrow f(x) = \frac{1}{x(\log x)^p}, \quad P > 0$$

Given series is non-negative and decreasing for $x \geq 2$ and $P > 0$.

When $P = 1$, Take $I_n = \int_2^n \frac{1}{x(\log x)}\,dx$

$$= [\log \log x]_2^n$$

$$= (\log \log n - \log \log 2)$$

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty}(\log \log n - \log \log 2) = \infty$$

So $\int_2^{\infty} f(x)\,dx$ divergent. By Integral test given series is divergent.

When $P \neq 1$.

$$In = \int_2^n f(x)\,dx = \int_2^n \frac{1}{x(\log x)^P}$$

$$= \int_2^n \frac{(\log x)^{-P}}{x}\ dx$$

take log x = t

$$\frac{1}{x}dx = dt$$

when x = z

then t = log z

when x = n then t = log n

So, $\text{In} = \int_{\log 2}^{\log n} (t)^{-P}\,dt$

$$= \left(\frac{t^{1-P}}{1-P}\right)_{\log 2}^{\log n}$$

$$= \frac{1}{1-P}\left[(\log n)^{1-P} - (\log 2)^{1-P}\right] \qquad \text{---(1)}$$

Take log x = t $\quad \frac{1}{x}\ dx = dt$

When x = 2 $\quad$ t = log 2

When x = n $\quad$ t = log n.

Now we consider two sub cases

(a) When P < 1 , we have

$$\text{In} = \frac{1}{1\text{-}P}\left[\lim_{n\to\infty}(\log n)^{1\text{-}P} - (\log 2)^{1\text{-}P}\right] \qquad \text{From (1)}$$

$$\lim_{n\to\infty} \text{In} = \frac{1}{1\cdot P}\left[\lim_{n\to\infty}(\log n)^{1-P} - (\log 2)^{1-P}\right]$$

$$= \frac{1}{1\cdot P}\cdot\infty = \infty$$

So, $\int_2^\infty f(x)\,dx$ is divergent. By integral test

Given series is divergent.

(b) When P > 1

$$In = \frac{1}{1-P}\left[(\log n)^{1-P} - (\log 2)^{1-p}\right] \qquad \text{From (1)}$$

$$\lim_{n\to\infty} In = \frac{1}{1-P}\left[\lim\ (\log n)^{1-P} - (\log 2)^{1-p}\right]$$

$$= \frac{1}{1-P}\left[0 - (\log 2)^{1-P}\right]$$

$$= \frac{(\log 2)^{1-P}}{P-1} \qquad [\because\ 1-P<0]$$

= Finite

So, $\int_2^\infty f(x)\,dx$ Convergent. By integral test given series is convergent. Hence given series

$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $P \rangle 0$ convergent if $P > 1$ and divergent if $0 < P \le 1$.

3. Show that the series (i) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ are convergent ?

Sol. :

(i) We test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence

$$f(n) = \frac{1}{n^2+1} \Rightarrow f(x) = \frac{1}{x^2+1}$$

f(x) is non-negative and monotonically decreasing function of x for $x \ge 1$

Take $In = \int_1^n f(x)\,dx$

$$= \int_1^n \frac{1}{x^2+1}\,dx = \left(\tan^{-1}x\right)_1^n$$

$= \tan^{-1} n - \tan^{-1} 1$

$\text{In} = \tan^{-1} n - \dfrac{\pi}{4}$

$\lim \text{In} = \lim \tan^{-1} n - \dfrac{\pi}{4} = \dfrac{\pi}{2} - \dfrac{\pi}{4}$

$= \dfrac{\pi}{4}$ = finite.

So, $\int_1^\infty f(x)$ convergent. By Integral test given series is also convergent.

(ii) Given $\sum_{n=1}^{\infty} \dfrac{1}{n(n+1)}$

Take $f(n) = \dfrac{1}{n(n+1)}$

$\Rightarrow f(x) = \dfrac{1}{x(x+1)}$

Take $\text{In} = \int_1^n f(x)\, dx$

$= \int_1^n \dfrac{1}{x(x+1)} dx = \int_1^n \left[\dfrac{1}{x} - \dfrac{1}{x+1}\right] dx$

$= \left[\log x - \log(1 + x)\right]_1^x$

$= \log n - \log(1 + n) + \log 2$

$\lim\limits_{n\to\infty} \text{In} = \lim\limits_{n\to\infty}\left[\log n - \log(1 + n)\right] + \log 2$

$= \lim \log \dfrac{n}{1 + n} + \log 2$

$= \lim \log \dfrac{1}{1 + \dfrac{1}{n}} + \log 2$

$= 0 + \log 2 = \log 2 =$ finite

So, $\int_1^\infty f(x)\,dx$ convergent. By integral test given series is convergent.

4. By integral test show that the series

(i) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2}$

are convergent.

Sol. :

(i) Given series is

$$\sum_{n=2}^{\infty}\left(\frac{1}{n\sqrt{n^2-1}}\right)$$

Take $f(n) = \frac{1}{n\sqrt{n^2-1}}$

$$\Rightarrow f(x) = \frac{1}{x\sqrt{x^2-1}}$$

$f(x)$ is positive and decreasing for $x \geq 2$

Now take $\text{In} = \int_2^n f(x)\,dx = \int_2^n \frac{1}{x\sqrt{x^2-1}}\,dx$

$$= \left(\text{Sec}^{-1}x\right)_2^n$$

$= (\text{Sec}^{-1} n - \text{Sec}^{-1} 2)$

$$\lim_{n\to\infty} \text{In} = \lim_{n\to\infty}\left[\text{Sec}^{-1}n - \text{Sec}^{-1}2\right]$$

$$= \lim_{n\to\infty} \text{Sec}^{-1}n - \text{Sec}^{-1}2 = \frac{\pi}{2} - \frac{\pi}{3}$$

$$= \frac{\pi}{6} \text{ Finite}$$

So $\int_2^\infty f(x)\,dx$ is convergent. By integral test given series is also convergent.

(ii) Given $\sum_{n=1}^{\infty} \frac{1}{\left(n^2+1\right)^2}$

Let $f(n)=\frac{1}{\left(n^2+1\right)^2} \Rightarrow f(x)=\frac{1}{\left(x^2+1\right)^2}$

f(x) is positive and decreasing

Take $\text{In} = \int_1^n f(x)\,dx = \int_1^n \frac{x}{\left(x^2+1\right)^2}\,dx$

$$= -\frac{1}{2}\left(\frac{1}{x^2+1}\right)_1^n = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n^2+1}\right]$$

$$\lim_{n\to\infty} \text{In} = \lim_{n\to\infty} \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n^2+1}\right] = \frac{1}{4} = \text{Finite}$$

So, $\int_1^{\infty} f(x)\,dx$ convergent. By integral test given series is convergent.

EXERCISE (3A)

Test the following series for convergence

1. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots\ldots\ldots + \frac{1}{n\cdot(n+1)} + \ldots\ldots$

2. $\cos\frac{\pi}{2} + \cos\frac{\pi}{4} + \cos\frac{\pi}{6} + \ldots\ldots\ldots + \cos\frac{\pi}{2n} + \ldots\ldots$

3. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \ldots\ldots\ldots$

4. $\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \ldots\ldots\ldots$

5. $a + b + a^2 + b^2 + a^3 + b^3 + \ldots\ldots\ldots$

6. $\sum_{n=1}^{\infty}\left[\frac{2^n+3^n}{6^n}\right]$

7. $\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\ldots\ldots$

8. $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+\ldots\ldots$

9. $\frac{1^2.2^2}{\angle 1}+\frac{2^2.3^2}{\angle 2}+\frac{3^2.4^2}{\angle 3}+\ldots\ldots..$

10. Test the convergence of the series

(i) $\sum\left[\frac{\sqrt{n+1}-\sqrt{n-1}}{n}\right]$ (ii) $\sum\left[\frac{\sqrt{n+1}-\sqrt{n}}{n^p}\right]$

(iii) $\sum\left[\frac{1}{(\log n)^{\log n}}\right]$ (iv) $\sum\left[\sqrt{n^2+1}-\sqrt{n^2-1}\right]$

(v) $\sum\left[\frac{\sqrt{n^2-1}}{\sqrt{n^3+1}}\right]$ (vi) $\sum\left[\sqrt{\frac{n}{2+3n^3}}\right]$

11. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots..$

12. Test the convergence of the series

(i) $\sum\left[\frac{\log n}{(\log n+1)}\right]^{n^2\log n}$ (ii) $\sum\left[\frac{n}{n+1}\right]^{n^2}$

(iii) $\sum\left[\frac{(n+1)^{n+1}}{n^n+1}-\frac{n+1}{n}\right]^{-n}$

13. $\frac{2}{1^2+1}+\frac{2^2}{2^2+1}+\frac{2^3}{3^2+1}+\ldots\ldots.$

14. $\frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9} + \ldots\ldots$

15. $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots.$

16. $\frac{x^2}{2\sqrt{1}} + \frac{x^3}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \ldots\ldots., \quad x \rangle 0$

17. $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots\ldots.$

18. $\frac{a}{3} + \frac{a^2}{6} + \frac{a^3}{11} + \ldots\ldots. + \frac{a^n}{n^2+2} + \ldots\ldots, \quad (a \rangle 2)$

19. (i) $\sum \left[\frac{n^2-1}{n^2+1} . x^n \right]$

(ii) $\sum \left[\sqrt{\frac{n+1}{n^3+1}} \cdot x^n \right]$

(iii) $\sum \left[\frac{x^n}{a+\sqrt{n}} \right]$

(iv) $\sum \left[\frac{1}{n} \right]^{\frac{1}{n}}$

20. $\frac{x}{\angle 1} + \frac{x^3}{\angle 3} + \frac{x^5}{\angle 5} + \ldots\ldots., \quad (x \rangle 0)$

21. Test the convergence of the series

(i) $\sum n\, e^{-n^2}$ (ii) $\sum \frac{1}{(n+1)\log(n+1)}$

(iii) $\sum \frac{1}{n(\log n)^3}$

ANSWERS EXERCISE (3A)

1. Convergent 2. Divergent

3. Divergent 4. Convergent

5. $\sum(u_n + v_n)$ converge only when |a| and |b| <

Where $\sum u_n = a + a^2 + a^3 + ...\infty$

$\sum v_n = b + b^2 + b^3 + ...\infty$

6. Convergent 7. Convergent

8. Divergent 9. Convergent

10. (i) Convergent (ii) Convergent $P \rangle \frac{1}{2}$

and divergent $P \le \frac{1}{2}$

(iii) Convergent (iv) divergent.

(v) Convergent (vi) divergent

11. Convergent

12. (i) Convergent (ii) Convergent

(iii) Convergent

13. divergent 14. Convergent

15. Convergent if $x < 1$ and divergent if $x \ge 1$

16. Convergent if $x \le 1$ and divergent if $x > 1$

17. Convergent if $x \le 1$ and divergent if $x > 1$

18. Convergent if $0 < a \le 1$ and divergent if $a > 1$

19. (i) Convergent if $x < 1$ and divergent if $x \ge 1$

(ii) Convergent if $x < 1$ and divergent if $x \ge 1$

(iii) Convergent if $x < 1$ and divergent if $x \ge 1$

(iv) Divergent.

20. Convergent.

21. (i) Convergent. (ii) Divergent

(iii) Convergent

3.9. Alternating Series :

A series with alternatively positive and negative terms is said to be alternating series.

Thus the infinite series of the form

$$\sum(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \ldots\ldots + (-1)^{n-1}u_n + \ldots.$$

Where $u_n > 0 \quad \forall$ n is an alternating series.

3.10. Leibnitz test :

An alternating series $u_1 - u_2 + u_3 - u_4 + \ldots. + (-1)^{n-1} u_n + \ldots..$ with $u_n > 0 \quad \forall$ n is convergent if

(a) $u_{n+1} \le u_n \quad \forall$ n (b) $\lim u_n = 0$

Proof :

Let S_n is the n^{th} partial Sum of alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots$ We shall show that $< S_{2n} >$ is convergent.

First we shall show $< S_{2n} >$ is bounded above.

$S_{2n} = u_1 - u_2 + u_3 - u_4 + \ldots. + u_{2n-1} - u_{2n}$

$= u_1 - \{[u_2 - u_3] + [u_4 - u_5] + \ldots\ldots + [u_{2n} - u_{2n-1}] + u_{2n}\}$

$\Rightarrow S_{2n} \le u_1 \qquad [\because u_2 \ge u_3, u_4 \ge u_5, \ldots.. u_{2n} \ge u_{2n-1}]$

$\Rightarrow S_{2n}$ is bounded above

Now, $S_{2n+2} = S_{2n} + u_{2n+1} - u_{2n+2}$

$\Rightarrow S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2}$

$\Rightarrow S_{2n+2} - S_{2n} \ge 0 \quad [\because u_n \ge u_{n+1} \ \forall \ n \Rightarrow u_{2n+1} \ge u_{2n+2}]$

$\Rightarrow < S_{2n} >$ is monotonically increasing

Since $< S_{2n} >$ is bounded above and monotonically increasing. So it is convergent if this sequence converges to s then $\lim S_{2n} = S$.

Now, $S_{2n+1} = S_{2n} + u_{2n+1}$

$= \lim S_{2n} + \lim u_{2n+1}$

$= S + 0 \qquad [\because \lim u_n = 0]$

$= S$

From above both the subsequence $< S_{2n} >$ and $< S_{2n+1} >$ converges to S.

$\Rightarrow$ to each $\in \ > 0 \ \exists$ + ve integer m' & m^* s.t.

$| S_{2n+1} - S | < \in \quad \forall \ 2n + 1 > m'$

& $| S_{2n} - S | < \in \quad \forall \ 2n > m^*$

$\Rightarrow | S_n - S | < \in \quad \forall \ n > m$

If $m = \max(m', m^*)$ then

$| S_{2n+1} - S | < \in \ \forall \ 2n+1 > m$

and $| S_{2n} - S | < \in \forall \ n > m$

$\Rightarrow |S_n - S| < \in \quad \forall\ n > m$

$\Rightarrow < S_n >$ converges to S

Hence $\sum(-1)^{n-1}u_n$ is convergent.

1. Show that the alternating series $\sum(-1)^{n-1}\frac{1}{n}$ is convergent?

Sol. :

The given series is $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots\ldots$

$u_n = \frac{1}{n}, \quad u_{n+1} = \frac{1}{n+1}$

$u_n - u_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \rangle\ 0 \quad \forall n$

$\Rightarrow u_n > u_{n+1} \quad \forall\ n$

$\lim u_n = \lim\frac{1}{n} = 0$

Hence by Leibnitz test given series is convergent.

2. Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-P},\ P \rangle\ 0$ is convergent?

Sol. :

Take $u_n = n^{-P}, P > 0$

$u_n = \frac{1}{n^P}$

$n+1 \rangle\ n \Rightarrow (n+1)^P \rangle\ n^P \Rightarrow \frac{1}{(n+1)^P} \langle\ \frac{1}{n^P} \quad \forall n$

$\Rightarrow u_{n+1} < u_n \quad \forall\ n$

$\lim u_n = \lim\frac{1}{n^P} = 0$

By Leibnitz test given series is convergent.

3. Show that the series $\sum (-)^{n-1} \frac{1}{x+(n-1)a}$, x > 0, a > 0 is convergent.

Sol.:

$$u_n = \frac{1}{x+(n-1)a}, \quad x \rangle 0, \quad a \rangle 0$$

$$u_{n+1} = \frac{1}{x + na}$$

$$u_n - u_{n+1} = \frac{1}{x+(n-1)\,a} - \frac{1}{x+na} = \frac{x+na-x-na+a}{(x+na)\,(x+(n-1)\,a)}$$

$$= \frac{a}{(x+na)\,(x+(n-1)\,a)} \rangle 0$$

$$\Rightarrow u_n \rangle u_{n+1} \quad \forall_n$$

$$\lim u_n = \lim \frac{1}{x+(n-1)\,a} = 0$$

By Leibnitz test given series is convergent.

3.11. Absolute convergence and conditional convergence :

A series $\sum u_n$ is called absolutely convergent if the series $\sum |u_n|$ is convergent.

4. the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} ....$ **is absolutely convergent because the series** $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} ....$ **is convergent.**

A series $\sum u_n$ is called conditionally (or semi or non-absolutely) convergent if the series $\sum u_n$

(a) Convergent

(b) Not absolutely i.e. $\sum |u_n|$ is divergent.

5. $\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots\ldots\ldots$ **is conditional convergent.**

Theorem 10: Every absolutely convergent series is convergent but converse need not be true.

Proof :

Let $\sum a_n$ is absolutely convergent this implies $\sum |a_n|$ is convergent. Since $|a_n|$ is convergent so to each $\in > 0$.

$\exists$ a + ve integer m s.t.

$| |a_{m+1}| + |a_{m+2}| + \ldots\ldots\ldots + |a_n| | < \in \quad \forall \; n < m$

$\Rightarrow |a_{m+1}| + |a_{m+2}| + \ldots\ldots\ldots + |a_n| < \in \quad \forall \; n < m$

We have $|a + b| \leq |a| + |b|$

So,

$|a_{m+1} + a_{m+2} + \ldots\ldots\ldots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \ldots\ldots\ldots + |a_n| < \in \;\; \forall \; n > m$

$\Rightarrow |a_{m+1} + a_{m+2} + \ldots\ldots\ldots + a_n| < \in \quad \forall \; n > m$

Hence by Cauchy general principle $\sum u_n$ is convergent.

Converse of the above theorem is not true. We take the $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ this series is convergent let $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ i.e. $\sum |u_n|$ is divergent.

Theorem 11: For the absolutely convergent series $\sum u_n$, the series of its + ve terms and the series of its – ve terms both are convergent.

Proof :

We consider Sn is the n^th partial Sum of $\sum u_n$ & S_n^1 is the n^th partial Sum of $\sum |u_n|$. If H_n and $-G_n$ are the Sum of the + ve and – ve terms in Sn.

$Sn = Hn - Gn$

and $S_n^1 = H_n + Gn$

These gives $H_n = \frac{S_n^1 + S_n}{2}$

and $\qquad G_n = \dfrac{S_n^1 - S_n}{2} \qquad$ ---(1)

Now, Both the series $\sum u_n$ and $\sum |u_n|$ are convergent because $\sum u_n$ is absolutely convergent. So sequence $<S_n>$ and $\langle S_n^1 \rangle$ are convergent.

Take lim $S_n = t_1$ and lim $S_n^1 = t_2$

Taking lim as $n \to \infty$ of (1) we have

lim H_n = ½ [lim S_n^1 + lim S_n] = ½ $(t_2 + t_1)$

& lim G_n = ½ [lim S_n^1 – lim S_n] = ½ $(t_2 - t_1)$

So, the sequence $<H_n>$ & $<G_n>$ are convergent.

Hence the series of + ve terms and the series of – ve terms are both convergent.

3.12. Rearrangement of series :

We consider a function f (domain I^+ and Range I^+) is one to one on I^+ if $\sum u_n$ and $\sum \nu_n$ are two series s.t. $\nu_n = u_{f(n)}$, n = 1, 2... Then $\sum \nu_n$ is rearrangement of $\sum u_n$.

1. $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$ is a series then the series $\sum \nu_n = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + ...$ is the rearrangement of $\sum u_n$.

Theorem 12: When the terms of an absolutely convergent series are rearranged the series remains convergent and its sum is not changed.

Proof :

We consider an absolutely convergent series $\sum a_n$. If we rearranging the terms of $\sum a_n$ we get the series $\sum b_n$ so that Every a in some b and Every b in some a.

When G_n is negative then $a_n + |a_n|$ is 0 and when a_n is positive then $a_n + |a_n|$ is $2a_n$

So, $0 \le a_n + |a_n| \le 2\,|a_n|$, $\quad \forall$ n

$\Rightarrow$ each term of $\sum(a_n + |a_n|)$ is non-negative and $\le$ the corresponding term of series $\sum(2\,|a_n|)$ Since $\sum(|a_n|)$ is convergent So $\sum(2\,|a_n|)$ is also convergent.

Hence the series $\sum(a_n + |a_n|)$ is convergent (comparison test)

Consider $\sum(|a_n|) = S$ & $\sum(a_n + |a_n|) = S'$

$\Rightarrow \sum a_n = S' - S \qquad (1)$

Both the series $\sum|a_n|$ and $\sum(a_n + |a_n|)$ have positive terms, they are not a affected by rearrangement. Therefore $\sum|a_n| = \sum|b_n| = S$

and $\sum(a_n + |a_n|) = S' = \sum(b_n + |b_n|)$

$\therefore \sum b_n = S' - S \qquad$ ---(2)

(1) & (2) gives

$\sum a_n = \sum b_n = S' - S$

Hence when the terms of an absolutely convergent series are rearranged the series remains convergent and its sum is not changed.

3.13.Riemann Rearrangement theorem :

Let $\sum u_n$ is a conditionally convergent series. Then $\sum u_n$ can be made converge to a no. α or diverge to $+\infty$ or $-\infty$ or oscillate finitely or infinitely by appropriate rearrangement of terms.

Proof :

We consider (i) $\alpha_n = u_n$ when $u_n \geq 0$ &
0 when $u_n < 0$ (ii) $\beta_n = -u_n$ when $u_n < 0$ and
0 when $u_n \geq 0$
clearly $u_n = \alpha_n - \beta_n$, $|u_n| = |\alpha_n + \beta_n|$ ---(1)

at least on of $\sum \alpha_n$, $\sum \beta_n$ divergent because $\sum u_n$ is conditionally convergent so $\sum|u_n|$ is divergent. Again from both the series $\sum \alpha_n$, $\sum \beta_n$ are converges together or diverge together because $\sum u_n$ is convergent.

We conclude $\sum \alpha_n$, $\sum \beta_n$ are diverge. Since $\lim u_n = 0$,

So $\lim \alpha_n = \lim \beta_n = 0$

(a) Consider the minimum no. of terms of $\sum \alpha_n$ be n_1 s.t. $\alpha_1 + \alpha_2 + ... + \alpha_{n1} > \alpha$

Now consider the minimum no. of terms of $\sum \beta_n$ be m_1 s.t.

$\alpha_1 + \alpha_2 + ... + \alpha_{n1} - \beta_1 + \beta_2 + ... + \beta_{m_1} < \alpha$.

Again consider the minimum no of next terms of $\sum \alpha_n$, n_2 (following α_{n1}) s.t.

$\alpha_1 + \alpha_2 +....+ \alpha_{n1} - \beta_1 - \beta_2 + ...- \beta_{m_1} + \alpha_{n_{1+1}} + \alpha_{n_{1+2}} +..... \alpha_{n_{1+n}} > \alpha$

Consider the minimum no. of next term of $\sum \beta_n$, m_2 s.t.

$\alpha_1 +....+ \alpha_{n1} - \beta_1 - \beta_2....-\beta_{m_1} + \alpha_{n_{1+1}} + \alpha_{n_1+n_2} - \beta_{m_{1+1}} ..... \beta_{m_{1+2}} \ \beta_{m_1+m_2}$

above process continue indefinitely.

If $\sum v_n$ is rearrangement series of $\sum u_n$ and $<S_n>$ its sequence of Partial Sum.

We have $S_{n_1} > \alpha$, $S_{n_1+m_1} < \alpha$..... we can easily show that $<S_n>$ converge to α .

So $\sum v_n$ converge to α.

(b) We consider

$\alpha_1 + \alpha_2 +....+ \alpha_m - \beta_1 + \alpha_{m_{1+1}} +....+ \alpha_{m_2} - \beta_2 + \alpha_{m_{2+1}} +.......$

It is rearrangement of $\sum u_n$, say $\sum v_n$. Its Partial Sum is G_n (say)

Now the Partial Sum of $\sum \alpha_n$ is unbounded. First we take m_1 so bigger s.t.

$\alpha_1 +......+ \alpha_{m_1} + 1+ \beta_1$

Then $m_2 > m_1$ so big s.t.

$\alpha_1 +......+ \alpha_{m_1} + \alpha_{m_{1+1}} + \alpha_{m_2} > 2 + b_1 + b_2$

Generally $m_n > m_{n-1}$ so big s.t.

$\alpha_1 +......+ \alpha_{m_n} > n +\beta_1+.....+ \beta_n, \quad n \in N$

Since Every Partial Sum in $G_{m_{1+1}}$, $G_{m_{2+2}}$,.....of $\sum v_n$ with last negative term $- \beta_n$ is greater than $n \in N$, so these partial sum are not bounded above. Consequently $\sum v_n$ diverge to $+ \infty$.

It we take the rearrangement $-\beta_1-\beta_2 \ldots\ldots -\beta_{m_1}+\alpha_1-\beta_{m_{1+1}}-\beta_{m_{1+2}}-\beta_{m_2}+\alpha_2-\beta_{m_{2+1}}\ldots\ldots$ like wise above we can show that rearrangement diverge to $-\infty$.
For other cases we consider the proper rearrangement and can be proved easily similarly.

EXAMPLE

1. Test the series $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\ldots\ldots$ **for convergence. If we rearrange the terms of the above series we get two series.**

$1+\frac{1}{2}-1+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+\frac{1}{5}\ldots\ldots$ **and** $1+\frac{1}{2}+\frac{1}{3}-1+\frac{1}{4}+\frac{1}{5}-\frac{1}{2}\ldots\ldots$

Find their Sum ?

Sol. :

Given, $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\ldots\ldots$

Since we have $S_{2n}=0 \Rightarrow \lim S_{2n}=0$

$S_{2n+1}=\frac{1}{n+1} \quad \Rightarrow \lim S_{2n+1}=\lim\frac{1}{n+1}=0$

$S_{2n+2}=0 \Rightarrow \lim S_{2n+2}=0$

Sum of the series tends to 0 as n tending to ∞.

So the given series is convergent.

II nd Part :

Let S_1 and S_2 are the Sum of first and second rearrange series of given series respectively.

$$S_1=1+\frac{1}{2}-1+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+\frac{1}{5}+\frac{1}{6}-\frac{1}{3}+\ldots\ldots$$

$$=\left[1+\left(\frac{1}{2}-1\right)\right]+\left[\frac{1}{3}+\left(\frac{1}{4}-\frac{1}{2}\right)\right]+\left[\frac{1}{5}+\left(\frac{1}{6}-\frac{1}{3}\right)\right]+\ldots..$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots..$$

$$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots..$$

$= \log(1+1) = \log 2$

$$S_2 = \lim_{n\to\infty}\left[\left(1 - 1 + \frac{1}{2} - \frac{1}{2} + \ldots\ldots + \frac{1}{n} - \frac{1}{n}\right) + \left[\sum_{i=1}^{2n} \frac{1}{n+i}\right]\right]$$

$$= \lim\left[0 + \sum_{i=1}^{2n}\left(\frac{1}{n+i}\right)\right]$$

$$= \lim \sum_{i=1}^{2n}\left(\frac{1}{n+i}\right) = \lim \sum_{i=1}^{2n}\left[\frac{\frac{1}{n}}{1+\frac{i}{n}}\right] = \int_0^2 \frac{dx}{1+x}$$

$= \log 3.$

EXAMPLE

1. Is the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ **is conditionally convergent ?**

Sol. :

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \ldots\ldots$$

$$u_n = \frac{1}{\sqrt{n}}, \; u_{n+1} = \frac{1}{\sqrt{n+1}}$$

$$u_n - u_{n+1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}\sqrt{n+1}} \rangle 0$$

$$u_n \rangle u_{n+1} \quad \forall n$$

$$\lim u_n = \lim \frac{1}{\sqrt{n}} = 0$$

By Leibnitz test given series is convergent.

Again we take the series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots..$$

$$= \sum \frac{1}{n^{\frac{1}{2}}}$$

= divergent as $P = \frac{1}{2} \langle 1$

So given series is conditionally convergent.

2. Show that the following series is not conditionally convergent

(i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots...$

(ii) $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \ldots..$

Sol. :

(i) We have $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots...$

Here $u_n = \frac{1}{2n-1}, u_{n+1} = \frac{1}{2n+1}$

$2n + 1 > 2n - 1 \quad \forall\ n$

$\Rightarrow \frac{1}{2n+1} \langle \frac{1}{2n-1}$

$\Rightarrow u_{n+1} < u_n \quad \forall\ n$

$\lim u_n = \lim \frac{1}{2n-1} = 0$

By Leibnitz test given series is convergent.

Now take $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots....$

$|u_n| = \frac{1}{2n-1}$

$$\nu_n = \frac{1}{n}$$

$$\lim \frac{|u_n|}{\nu_n} = \lim \frac{n}{2n-1} = \lim \frac{1}{2-\frac{1}{n}} = \frac{1}{2}$$

= Finite and non-zero

So, by comparison test both the series $\sum |u_n|$ & $\sum \nu_n$ converge or diverge together.

Since $\sum \nu_n = \sum \frac{1}{n}$ is divergent as P = 1 so the series $\sum |u_n|$ is divergent.

Hence the given series is conditionally convergent.

(ii) We have to show the series

$\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \ldots\ldots$ is conditionally convergent.

$$u_n = \frac{1}{\sqrt{n+1}+1}, \quad u_{n+1} = \frac{1}{\sqrt{n+2}\;+\;1}$$

Since, $\left(\sqrt{n+1}+1\right) < \left(\sqrt{n+2}\;+\;1\right) \quad \forall n$

So, $\frac{1}{\sqrt{n+1}+1} > \frac{1}{\sqrt{n+2}\;+\;1} \quad \forall n$

$\Rightarrow u_n > u_{n+1} \quad \forall\, n$

$$\lim\, u_n = \lim \frac{1}{\sqrt{n+1}+1} = 0$$

By Leibnitz test given series is convergent

Now take, $\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} + \ldots\ldots$

$$|u_n| = \frac{1}{\sqrt{n+1}+1}$$

Take $\nu_n = \frac{1}{\sqrt{n}}$

$$\lim \frac{|u_n|}{v_n} = \lim \frac{\sqrt{n}}{\sqrt{n+1}+1} = 1 \text{ Finite and non-zero}$$

So, by comparison test both the series $\sum |u_n|$ & $\sum v_n$ are convergent or divergent together. Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$ is divergent as $P = \frac{1}{2} \langle 1$ so the series $\sum |u_n|$ is divergent.

Hence the given series is conditionally convergent.

3. Examine the series $\sum (-1)^n \left[\sqrt{n^2+} - n\right]$ for absolutely convergence ?

Sol. :

Given series is $\sum (-1)^n \left[\sqrt{n^2+1} - n\right]$

$$u_n = \left[\sqrt{n^2+1} - n\right]$$

$$u_n = \frac{\left(\sqrt{n^2+1} - n\right)}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}-n} = \frac{1}{\sqrt{n^2+1}-n}$$

$$u_{n+1} = \frac{1}{\sqrt{(n+1)^2+1}+(n+1)}$$

Since, $\left[\sqrt{(n+1)^2+1}+(n+1)\right] \rangle \left[\sqrt{n^2+1} + n\right] \quad \forall n$

$$\Rightarrow \frac{1}{\sqrt{(n+1)^2+1}+(n+1)} \langle \frac{1}{\sqrt{(n^2+1)} + n}$$

$$\Rightarrow u_{n+1} < u_n \qquad \forall\, n$$

$$\lim u_n = \lim \frac{1}{\sqrt{n^2+1} + n} = 0$$

By Leibnitz test the given series is convergent.

Now take the series $\sum|u_n| = \sum\left[\sqrt{n^2+1} - n\right]$

$$|u_n| = \frac{1}{\sqrt{n^2+1} + n}$$

Consider $v_n = \dfrac{1}{n}$

$$\lim \frac{|u_n|}{v_n} = \lim \frac{n}{\sqrt{n^2+1} + n} = \lim \frac{1}{\sqrt{\frac{n^2+1}{n^2}} + 1}$$

= ½ = Finite and non-zero.

By Comparison test $\sum|u_n|$ & $\sum v_n$ both series are converge or diverge together. Since the series $\sum v_n = \sum \frac{1}{n}$ is divergent so the series $\sum|u_n|$ is divergent. Hence the given series is conditionally convergent.

4. Show that the series $1 - \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}}$........ **is absolutely convergent ?**

Sol. :

We have $\sum(-1)^{n-1}\dfrac{1}{n\sqrt{n}} = 1 - \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} - \dfrac{1}{4\sqrt{4}} + ...$

$$u_n = \frac{1}{n\sqrt{n}}, \quad u_{n+1} = \frac{1}{(n+1)\sqrt{n+1}}$$

$\Rightarrow u_{n+1} < u_n \quad \forall\ n$

$$\lim u_n = \lim \frac{1}{n\sqrt{n}} = 0$$

By Leibnitz test given series is convergent.

Take the series $\sum|u_n| = 1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + ...$

$$|u_n| = \frac{1}{n\sqrt{n}} = \frac{1}{n^{1+\frac{1}{2}}} = \frac{1}{(n)^{\frac{3}{2}}}$$

$$\sum |u_n| = \sum \frac{1}{n^{3/2}} \quad \text{is convergent}$$

as $P = \frac{3}{2} \rangle 1$

Hence the given series is absolutely convergent.

EXERCISE (3B)

Test the following series for convergence

1. $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + ...$

2. $\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...$

3. $\log\left(\frac{1}{2}\right) - \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) - \log\left(\frac{4}{5}\right) + ...$

4. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...$

5. $\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} ....$

6. $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} - \frac{1}{7 \cdot 8} + ...$

7. $\sum_{n=1}^{\infty} (-1)^n \frac{\left[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right]}{n}$

Test the following series for absolutely convergence or conditionally convergence.

8. $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} - \frac{1}{7 \cdot 8} + ...$

9. $1 - \frac{1}{\angle 3} + \frac{1}{\angle 5} - \frac{1}{\angle 7} + ...$

10. (i) $\sum (-1)^{n-1} \text{Sin}\frac{1}{n}$ (ii) $\sum (-1)^{n-1} \frac{n}{2n-1}$

(iii) $\sum (-1)^{n+1} \frac{n}{n^2+1}$ (iii) $\sum (-1)^{n-1} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$

11. $1 - \frac{1}{2^3}(1+2) + \frac{1}{3^3}(1+2+3).......$

12. $1 - \frac{1}{4 \cdot 3} + \frac{1}{4^2 \cdot 5} \cdot \frac{1}{4^3 \cdot 7} + ...$

13. $1 + \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + ...$ for all values of x.

14. Prove that the series

$2 \sin \frac{x}{3} + 4 \sin \frac{x}{9} + 8 \sin \frac{x}{27} + ...$ converge absolutely for all finite values of x.

15. Show that the series

$1 - 2 + 3 - 4 + 5 - 6 + ...$ oscillates finitely.

16. Prearrangement the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}........$ to reduce its sum to zero.

17. Prove that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \log 2$

18. Show that the rearrangement

$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + ...$ of the convergent series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ .... does not converge to the same limit ?

19. Prove that the series $\sum \left(\frac{\sin n\theta}{n^2}\right)$ is absolutely convergent ?

20. Prove if $\sum a_n$ converges and if $\sum(b_n - b_{n+1})$ converges absolutely then $\sum a_n b_n$ converges ?

21. Test the convergence and absolutely convergence of the series $\sum(-1)^{n-1} Sin\frac{1}{n}$?

ANSWERS EXERCISE (3B)

1. Convergent
2. Convergent
3. Convergent
4. Convergent
5. Convergent
6. Convergent
7. Convergent
8. Absolutely Convergent
9. Absolutely Convergent
10. (i) Conditionally Convergent
    (ii) Not Convergent
    (iii) Conditionally Convergent
    (iv) absolutely Convergent
11. Conditionally Convergent
12. Absolutely Convergent
13. Absolutely Convergent
14. Conditionally Convergent.

***

Chapter 4

POWER SERIES

4.1. Defination

A series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$ or $\sum_{n=0}^{\infty} a_n z^n$ is called power series where z is complex variable and a, a_n are complex constant.

4.2. Absolute convergent

A power series $\sum_{n=0}^{\infty} a_n z^n$ is called absolute convergent if $\Sigma |a_n||z^n|$ is convergent.

4.3. Conditional convergent

A power series $\Sigma a_n z^n$ is called conditional convergent or semi convergent if $\Sigma a_n z^n$ is convergent but $\Sigma |a_n||z^n|$ is not convergent.

Theorem 1. The power series $\Sigma a_n z^n$ either

(i) Convergence for every z

(ii) Convergence only for z = 0

(iii) Convergence for sum value of z.

Proof : It is sufficient to produce an example in each case

(i) Consider the series $\Sigma \frac{z^n}{\angle n}$

Let $\Sigma u_n (z) = \Sigma \frac{z^n}{\angle n}$

$$\therefore \qquad \Sigma u_{n+1}(z) = \Sigma \frac{z^{n+1}}{\angle n+1}$$

Now by D Alembertis ratio test

$$\lim_{n\to\infty}\left|\frac{u_n}{u_n+1}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| = \infty$$

Hence the power series $\Sigma \frac{z^n}{\angle n}$ is convergent for every z.

(ii) Consider the power series $\sum z^n \angle n$

Let $\Sigma u_n = \Sigma z^n Ln$

Then $\lim_{n\to\infty}|u_n| = \lim_{n\to\infty} \angle n |z|^n$ $\begin{cases} 0 \text{ if } z=0 \\ \infty \text{ if } z\neq 0 \end{cases}$

$\therefore$ The given series is convergent for z = 0 and divergent for z $\neq$ 0

(iii) The geometric series $\sum_{n=1}^{\infty} z^n$ converges for $|z| < 1$ and diverge for $|z| \geq 1$.

Theorem 2. If the power series $\sum_{n=1}^{\infty} a_n z^n$ converse for particular values z_0 of z then it converges absolutely for every z for which $|z| < |z_0|$.

Proof : Suppose the power series $\sum_{n=1}^{\infty} a_n z^n$ is convergent for z = z_0 so that its n^{th} term must tend to zero as $n \to \infty$

i.e. $\lim_{n\to\infty} a_n z_0^{\,n}$

So we can find number m > 0 s.t

$$\left|a_n z_0{}^n\right| \le M \quad \forall n$$

Now $\quad \left|a_n z^n\right| \le \dfrac{M}{\left|z_0\right|^n}\left|z^n\right| = M\left|\dfrac{z}{z_0}\right|^n$

$\Rightarrow \quad \left|a_n z^n\right| \le M\left|\dfrac{z}{z_0}\right|^n$

But geometric series $\sum \dfrac{|z|^n}{\left|z_0\right|^n}$ is convergent for all z, s.t

$$\frac{|z|}{\left|z_0\right|} < 1 \quad \text{i.e} \quad |z| < \left|z_0\right|$$

Hence $\sum a_n z^n$ is absolutely convergent for all z for which $|z| < \left|z_0\right|$.

4.4. Radius of convergence of power series

Consider the power series $\sum a_n z^n = \sum u_n(z)$ then by cauchy root test we know that $\sum u_n(z)$ is convergent if

$$\lim_{n\to\infty}\left|u_n\right|^{\frac{1}{n}} < 1$$

$\Rightarrow \quad \lim\limits_{n\to\infty}\left|a_n z^n\right|^{\frac{1}{n}} < 1 \Rightarrow \lim\limits_{n\to\infty}\left|a_n\right|^{\frac{1}{n}}|z| < 1$

$\Rightarrow \quad |z|\lim\limits_{n\to\infty}\left|a_n\right|^{\frac{1}{n}} < 1$

$\Rightarrow \quad |z|\dfrac{1}{R} < 1$ $\qquad$ Where $\dfrac{1}{R} = \lim\limits_{n\to\infty}\left|a_n\right|^{\frac{1}{n}}$

$\Rightarrow \quad |z| < R$

Here R is called radius of convergent and circle $|z| = R$ is called circle of convergence with in which power series $\sum a_n z^n$ is convergent.

Hence If $|z| < R$ then series is convergent

and If $|z| > R$ then the series is divergent.

Note : Now if we draw a circle of radius R with centre at origin then

(i) The series $\sum a_n z^n$ is convergent for every z within the circle

(ii) The series $\sum a_n z^n$ is divergent for every z outside the circle.

This type circle is called circle of convergence and radius R is called radius of convergence of the power series $\sum a_n z^n$.

4.5. Important result for radius of convergence

(i) $$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

(ii) $$\frac{1}{R} = \lim_{n\to\infty}|a_n|^{\frac{1}{n}}$$

Remark :

(i) If R = 0 then series is convergent only when Z = 0

(ii) If R is finite then series is convergent at every point within circle and is divergent at every point outside the circle.

Theorem 3. To show that power series $\sum na_n z^{n-1}$ obtained by differentiating the power series $\sum a_n z^n$ has the same radius of convergence as the original series $\sum a_n z^n$.

Proof : Let R and R' be the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} n a_n z^{n-1}$ respectively, Now we shall prove that R = R'

Then by definition of radius of convergence we know that.

$$\frac{1}{R'} = \lim_{n\to\infty} |na_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} |a_n|^{\frac{1}{n}} \qquad \left\{\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right.$$

$$\frac{1}{R'} = \frac{1}{R}$$

or $\quad R' = R \qquad$ Hence proved.

5.6 Important test for convergence of series

(i) If $\sum u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$

(ii) $\sum u_n$ is absolutely convergent if

$|u_n| \le |v_n|$ and $\sum v_n$ is convergent $\qquad \{$By comparison test

(iii) If $\lim_{n\to\infty} |u_n|^{\frac{1}{n}} = \ell$ Then $\sum u_n$ is convergent if $\ell < 1$ and divergent $\ell > 1$ and test fail if $\ell = 1$ $\qquad \{$By root test

(iv) $\sum u_n$ is convergent if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ and divergent if $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$

$\{$By ratio test

EXAMPLE

1. Find the radius of convergence of the following series

(a) $\Sigma \frac{z^n}{n^n}$ **(b)** $\Sigma (4+3i)^n z^n$

Solution :

(a) Here $\Sigma a_n z^n = \Sigma \frac{z^n}{n^n}$

Then $a_n = \frac{1}{n^n}$

$$\therefore \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{1}{n^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Then $R = \frac{1}{0} = \infty$

(b) Here $\Sigma a_n z^n = \Sigma (4+3i)^n z^n$

Then $a_n = (4+3i)^n$

$$\therefore \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|(4+3i)^n\right|^{\frac{1}{n}}$$

$$= |4+3i| = \sqrt{16+9} = \sqrt{25} = 5$$

$$\therefore \quad R = \frac{1}{5}$$

2. Find the radius of convergence of the following series

(a) $\Sigma b_n^n z^n$ where $b_n = 1+1+\frac{1}{\angle 2}+\frac{1}{\angle 3}+...+\frac{1}{\angle n}$

(b) $\Sigma(\log n)^n z^n$

Solution :

(a) Here $\sum a_n z^n = \sum b_n^n z^n$

Then $a_n = b_n^n$

$$\therefore \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} |b_n^n|^{\frac{1}{n}} = \lim_{n\to\infty} b_n$$

$$= \lim_{n\to\infty}\left[1+1+\frac{1}{\angle 2}+\frac{1}{\angle 3}+\ldots+\frac{1}{\angle n}\right] = e$$

Then $R = \dfrac{1}{e}$

(b) Here $\sum a_n z^n = \Sigma(\log n)^n z^n$

Then $a_n = (\log n)^n$

$$\therefore \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|(\log n)^n\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \log n = \infty$$

$$\therefore \quad R = 0$$

3. Find the radius of convergence of the series $\Sigma\left(1+\dfrac{1}{n}\right)^{n^2} z^n$

Solution : Here $\sum a_n z^n = \Sigma\left(1+\dfrac{1}{n}\right)^{n^2} z^n$

Then $a_n = \left(1+\frac{1}{n}\right)^{n^2}$

$$\therefore \qquad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^{n^2}\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left[1+\frac{1}{n}\right]^n = e$$

Then $R = \frac{1}{e}$

4. Find the radius of convergence of the series

$$\frac{z}{2} + \frac{1.3}{2.5}z^2 + \frac{1.3.5}{2.5.8}z^3 + ...$$

Solution : Here $a_n = \frac{1.3.5}{2.5.8}....\frac{(2n-1)}{(3n-1)} =$

$$\therefore \qquad a_{n+1} = \frac{1.3.5...(2n-1)(2n+1)}{2.5.8...(3n-1)(3n+2)} =$$

$$\therefore \qquad \frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{2}{3}\left(\frac{1+\frac{1}{2n}}{1+\frac{2}{3n}}\right)$$

$$= \frac{2}{3}$$

Then $R = \frac{3}{2}$

5. Find the radius of convergence of the following series

(a) $\sum_{n=0}^{\infty} \frac{(\angle n)^2}{\angle 2n} z^n$ (b) $\Sigma \frac{2^{-n} z^n}{1+in^2}$

(c) $\Sigma \frac{(-1)^n (z-2i)^n}{n}$

Solution :

(a) Here $\Sigma a_n z^n = \sum_{n=0}^{\infty} \frac{(\angle n)^2}{\angle 2n} z^n$

Then $a_n = \frac{(\angle n)^2}{\angle 2n}$

$$a_{n+1} = \frac{(\angle n+1)^2}{\angle 2(n+1)} = \frac{[(n+1)\angle n]^2}{(2n+2)(2n+1)\angle 2n}$$

Now $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{(n+1)}{2(2n+1)}$

$\therefore$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{R} = \lim_{n\to\infty} \frac{(n+1)}{2(2n+1)} = \frac{1}{4}$

Then R = 4

(b) Here $\Sigma a_n z^n = \Sigma \frac{2^{-n} z^n}{1+in^2}$

Then $a_n = \frac{2^{-n}}{1+in^2}$

$$\therefore \qquad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{2^{-n}}{1+in^2} \right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left[\frac{2^{-n}}{\sqrt{1+n^4}} \right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2^{-1}}{\left(1+n^4\right)^{\frac{1}{2n}}}$$

$$= \lim_{n\to\infty} \frac{1}{2} \frac{1}{\left(n^2\right)^{\frac{1}{n}}} \left(1+\frac{1}{n^4}\right)^{\frac{-1}{2n}}$$

$$= \lim_{n\to\infty} \frac{1}{2} \frac{1}{\left(n^{\frac{1}{n}}\right)^2} \left[1 - \frac{1}{2n^5} + ...\right]$$

$$= \frac{1}{2} x 1 x 1 = \frac{1}{2} \qquad \left\{\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right.$$

(c) Here $\Sigma a_n (z-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (z-2i)^n}{n}$

Then $a_n = \frac{(-1)^n}{n}$

$$\therefore \qquad a_{n+1} = \frac{(-1)^{n+1}}{(n+1)}$$

$$\therefore \qquad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1$$

$\therefore$ R = 1 $\qquad \left\{\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1\right.$

Exercise : 4

1. Find the radius of convergence of the following power series

   (a) $\sum\left(1-\frac{1}{n}\right)^{n^2} z^n$ (b) $\sum\frac{z^n}{2^n+1}$

2. Find the radius of convergence of the following series

   (a) $\sum\angle nz^n$ (b) $\sum\frac{\angle n}{n^n}z^n$

3. Find the radius of convergence of the series $\sum\frac{n+1}{(n+2)(n+3)}z^n$

4. Find the radius of convergence of the series $\sum_{n=1}^{\infty}\frac{z^n}{\log n}$

6. If R_1 and R_2 are the radius of convergence of the power series $\sum a_n z^n$ and $\sum b_n z^n$ respectively, then show that the radius of convergence of the power sereis $\sum a_n b_n z^n$ is $R_1 R_2$.

7. Find the radius of convergence of the series $\sum\frac{2+in}{2^n}z^n$.

8. Prove that the series $\sum 2^{\sqrt{n}} z^n$ has unit radius of convergence.

9. Find the radius of convergence of the series $\sum\frac{z^n}{2^{n+1}}$.

Answer : 4

1. (a) R = e, (b) R = 2	2. (a) R = 0, (b) R = e
3. R = 1	4. R = 1
7. R = 2	8. R = 1

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Chapter 5

UNIFORM CONVERGENCE SEQUENCE AND SERIES OF THE FUNCTION

5.1. Introduction

In this chapter we confine out attention to uniform convergence of sequence and series of the function

Sequence- A : Sequence in the set x is a mapping of the set N of positive integer into X the image of S(n) of n denoted by S_n and written as $\{S_1, S_2, \ldots S_n\}$ or $\{S_n\}$.

5.2. Convergence of a sequence of a function

A sequence of the function $\{f_n(x)\}$ is defined on the set X is called cauchy sequence, if for any given $\varepsilon > 0 \ \exists \ n_0 \in N.S.t \ \forall m, n \geq n_0$

$$\Rightarrow \qquad |f_n(x) - f_m(x)| < \varepsilon \ \forall n \in X$$

5.3. Convergence of sequence

A sequence $\{f_n(x)\}$ is Said to be convergence to f if for any given

$$\varepsilon > 0 \ \exists \ n_0 \in N.S.t$$

$$\forall n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon \ \forall n \in X$$

5.4. Uniformly bounded sequence

A sequence $f_n(x)$ define on the set x is said to be uniformly bounded if $\exists$ positive real number M such that $|f_n(x)| \leq M \ \forall n$ and $\forall n \in X$

5.5. Point wise convergence sequence

A sequence $\{f_n(x)\}$ is said to be pointwise convergence on x to the function f(x) if

$$\lim_{n\to\infty} f_n(x) = f(x)\ \forall n \in X$$

in other word for any given $\varepsilon > 0\ \exists\, n_0 \in N$.S.t

$$\forall n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon\ \forall n \in X$$

5.6. Uniform convergence of a sequence

Suppose that the sequence $\{f_n(x)\}$ convergence for every $x \in X$ it means that $f_n(x)$ tend to definite limit $f(x)$ as $n \to \infty\ \forall\ x \in X$

i.e. $$\lim_{n\to\infty} f_n(x) = f(x)\ \forall\ x \in X$$

then from the definition of limit for any given $\in > 0\ \exists\, m \in N$.S.t

$$\forall n \geq m \Rightarrow |f_n(x) - f(x)| < \varepsilon\ \forall\ x \in X$$

Let m = m (ε, x) if we keep ε is fixed and vary x. Then we get the set of value of m for different $x \in X$. This set of values of m may or not may have an upper bound if this set has an upper bound n_0, Then

$$\forall n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon\ \forall x \in X$$

in such case $\{f_n(x)\}$ is said to be uniformly convergent

5.7. Definition

A sequence of function $\{f_n(x)\}$ defined on the set X is said to be uniformly convergent on x if given $\varepsilon > 0\ \exists\, n_0 \in N$.S.t

$$n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon\ \forall x \in X$$

<u>Note</u> :- The reader should noted that there is a fixed m for every x in case of uniform convergence whereas for pointwise convergence or ordinary convergence, one value of m will correspond to one value of X.

5.8. Point of non uniform convergence

A point $x = x_1$ is said to be non uniform convergence for sequence $\{f_n(x)\}$, if the sequence does not convergence uniformly in any neighborhood of x_1, however small.

<u>Remark</u> 1. It is clear that every uniformly convergent sequence is pointwise convergence.

<u>Remark</u> 2. Uniform limit function is same as pointwise limit function.

Uniform convergence of a series

A series $\sum_{n=1}^{\infty} u_n(x)$ is said to to convergence uniformly on x iff the sequence $\{f_n(x)\}$ converge uniformly on X in other word for any given

$$\varepsilon > 0 \quad \exists\ n_0 \in N \text{ s.t } \quad \forall n \geq n_0 \Rightarrow |S_n(x) - S(x)| < \varepsilon \ \ \forall \ \ x \in X$$

when S_n is the n^{th} partial sum of the series and s is the sum of function.

5.9. Cauchy's general principle of uniform convergence

Theories 1. A sequence $\{f_n(x)\}$ defined on the set X is uniformly convergent on X if and only if for any given $\varepsilon > 0 \quad \exists\ n_0 \in N$ s.t

$$m, n \geq n_0 \Rightarrow |f_m(x) - f_n(x)| < \varepsilon$$

Proof. The only if part:

Let the sequence $\{f_n(x)\}$ uniformly convergent on X then for any given $\varepsilon > 0 \quad \exists\ n_0 \in N$ s.t

$$n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall x \in X$$

hence if $n, m \geq n_0$ we get for any $x \in X$

$$|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) + f_n(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $\quad |f_m(x) - f_n(x)| < \varepsilon$ ....(1)

The if part: Suppose for only $\in > 0$ and $x \in X$, $\exists n_0 \in N$

S.t. $\quad n, m \geq n_0 \Rightarrow |f_m(x) - f_n(x)| < \varepsilon$ ....(2)

Now we shall show that $\{f_n(x)\}$ is uniformly convergent on X by (2) we know that $f_n(x)$ is cauchy sequence but every cauchy sequence is convergent.

So $\lim\limits_{n\to\infty} f_n(x)$ exist $\forall x \in X$

we define $\lim\limits_{n\to\infty} f_n(x) = f(x)\ \forall x \in X$

Keeping m fixed in (2) and letling $n \to \infty$ then we get

$$|f_m(x) - f(x)| < \in \quad \forall m \geq n, \quad x \in X$$

It follows that sequence $\{f_n(x)\}$ converges uniformly to f

Theorem 2. A sequence $\{f_n(x)\}$ on the set X is uniformly convergent iff any given $\varepsilon > 0\ \exists\ n_0 \in N$

S.t $\quad n \geq n_0 \Rightarrow |f_{n+p}(x) - f_n(x)| \in, \quad P \in N$

Proof : The only if part. Let sequence $f_n(x)$ be uniformly convergent on X so $\{f_n(x)\}$ convergent uniformly to f(x) $\forall x \in X$ then for any given $\varepsilon > 0\ \exists n_0 \in N$ s.t

$$n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad \forall\ x \in X$$

Now be proved that

$$|f_{n+p}(x) - f_n(x)| < \varepsilon, \quad \forall\ n \geq n_1 \text{ and } p \in N$$

If $\ n \geq n_0$ and $p \in N$ then $\quad \forall x \in X$

$$|f_{n+p}(x) - f_n(x)| = |f_{n+p}(x) - f(x) + f(x) - f_n(x)|$$

$$\leq |f_{n+p}(x) - f(x)| + |f(x).....f_n(x)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \quad \left|f_{n+p}(x) - f_n(x)\right| < \varepsilon \ \forall\ n \geq n_0 \text{ and } p \in N \ \forall x \in X \text{ ....(1)}$$

The if part : Suppose $\{f_n(x)\}$ is a sequence then for a given $\in > 0$ and $\exists n_0 \in N$ and $p \in N$

$$\text{s.t} \quad \left|f_{n+p}(x) - f_n(x)\right| < \varepsilon \text{ ....(2)}$$

then we show that $\{f_n(x)\}$ is uniformly convergent, Now put m = n + p in (2) we get

$$\left|f_m(x) - f_n(x)\right| < \varepsilon \quad \forall\ m, n \geq n_0 \text{ and } \forall x \in X \text{ ....(3)}$$

$\Rightarrow$ $\{f_n(x)\}$ is a cauchy sequence but every cauchy sequence is convergent so

$$\lim_{n\to\infty} f_n(x) \text{ exist} \quad \forall x \in X$$

we define $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X$

now keeping m fixed in (3) and letting n $\to \infty$ then we get

$$\left|f_m(x) - f(x)\right| < \varepsilon \quad \forall\ m \geq n_0 \text{ and } \forall x \in X$$

It follows that sequence $\{f_n(x)\}$ converges uniformly to f.

Theoram 3. A series $\sum u_n(x)$ converges uniformly on x iff for any given

$$\varepsilon > 0\ \exists n_0 \in N \quad \text{s.t}$$

$$n \geq n_0 = \left|u_{n+1}(x) + u_{n+2}(x) + \text{......} u_{n+p}(x)\right| < \varepsilon, \quad p = 1, 2 \ldots$$

and $\quad \forall x \in X$

Proof : Let $f_n(x) = \sum_{i=1}^{n} u_i(x)$

where $\sum u_n(x)$ is uniformly convergent on X iff the sequence $\{f_n(x)\}$ is uniformly convergent. Now let $\{f_n(x)\}$ is uniformly convergent on X iff for given $\varepsilon > 0\ \exists n_0 \in N \quad$ s.t

$$n \geq n_0 = \left|f_{n+p}(x) - f_n(x)\right| < \varepsilon, \quad \forall x \in X$$

which proof as in theorem (2)

$\therefore$ $\quad \sum u_n(x)$ is uniformly convergent iff

for given $\varepsilon > 0 \, \exists n_0 \in N$ s.t

$$n \geq n_0 = \left|f_{n+p}(x) - f_n(x)\right| < \varepsilon, \quad \forall\, p \in N, x \in X \text{ ....(2)}$$

Again now let $f_{n+p}(x) - f_n(x) = \sum_{i-1}^{n+p} u_i(x) - \sum_{i-1}^{n} u_i(x) = \sum_{i-n+1}^{n+p} u_i(x)$

$$= u_{n+1}(x) + u_{n+2}(x) + ...+ u_{n+p}(x) \text{ ....(3)}$$

with the help of (3), (2) becomes

$$n \geq n_0 \Rightarrow \left|u_{n+1}(x) - u_{n+2}(x) + ...u_{n+p}(x)\right|$$

hence proved.

5.10. Test for uniform convergence

Theorem 4. (M_n - test) Let $\{f_n(x)\}$ be a sequence of function define on the set X.

Let $\quad \lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in X$ and let

$Mn = \sup\left|f_n(x) - f(x)\right| \quad \forall x \in X$ then $f_n(x)$ converges uniformaly to f iff $\lim_{n\to\infty} M_n = 0$

Proof : The only if part. Let the sequence $\{f_n(x)\}$ converges uniformly to f on X then for any given $\in > 0 \, \exists n_0 \in N$ s.t

$$n \geq n_0 \Rightarrow \left|f_n(x) - f(x)\right| < \varepsilon, \quad \forall\, x \in X \text{ ....(1)}$$

Keeping n fixed and taking supremum of both sides.

For verifying x it follows $M_n < \varepsilon \quad \forall \quad n \geq n_1$

or $\quad |M_n - 0| < \varepsilon \quad \forall \quad n \geq n_1$

$\Rightarrow \quad \lim_{n\to\infty} M_n = 0 \text{ as } n \to \infty$

The if part : Let $M_n \to 0 \text{ as } n \to \infty$ then given

$$\varepsilon > 0 \, \exists n_0 \in N \quad \text{s.t}$$

$$n \geq n_0 = M_n < \varepsilon$$

But M_n is the supremum of $|f_n(x) - f(x)|$

for varying x

Hence $|f_n(x) - f(x)| \leq M_n < \varepsilon \quad \forall x \in X, n \geq n_1$

Hence $\{f_n(x)\}$ converges uniformly to f on X.

Theorm 5. (**Weierstrass's M-test**) - A series $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on X if ΣM_n is convergent series of positive constant s.t. $|u_n(x)| \leq M_n \ \forall n$ and $\forall x \in X$.

Proof : Let ΣM_n is convergent series of positive constant s.t. $|u_n(x)| \leq M_n \ \forall n$ and $\forall x \in X$.

Now we shall show that $\Sigma u_n(x)$ is uniformly convergent.

$\therefore$ For any given $\varepsilon > 0 \, \exists n_0$ s.t

$$n \geq n_0 = |M_{n+1} + M_{n+2} ...... + M_{n+p}| < \varepsilon, \quad \forall \ p \in N$$

also $\quad |u_n(x)| < M_n$

$$\therefore \quad \left|u_{n+1}(x)+u_{n+2}(x)+....+u_{n+p}(x)\right| \le \left|u_{n+1}(x)\right|\left|u_{n+2}(x)\right|+\left|u_{n+p}(x)\right|$$

$$\le M_{n+1}+M_{n+2}......+M_{n+p}$$

$$=\left|M_{n+1}+M_{n+2}......+M_{n+p}\right|<\in, \text{as } M_n \to 0$$

Then $\left|u_{n+1}(x)+u_{n+2}(x)......+u_{n+p}(x)\right|<\varepsilon, \quad \forall n \in n_1\ \forall x \in X$

Hence $u_n(x)$ is uniformly convergent.

EXAMPLE

1. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(nx)$ **converges uniformly on R.**

Solution : Here $u_n(x)=\frac{1}{n^2}\cos(nx)$ then

$$\left|u_n(x)\right|=\left|\frac{1}{n^2}\cos(nx)\right| \le \frac{1}{n^2}=M_n$$

Now $\left|u_n(x)\right| \le M_n, \Sigma M_n = \Sigma \frac{1}{n^2}$

by p series test p = 2 > 1 ΣM_n is convergent.

$\therefore \quad \Sigma u_n(x)$ is unformly convergent on R.

2. Show that $\sum_{1}^{\infty} \frac{\sin nx}{n^p}, p>1$ **convergent on R.**

Solution : Here $u_n(x)=\frac{\sin nx}{n^p}$ then

$$\left|u_n(x)\right| = \left|\frac{\sin nx}{n^p}\right| \le \frac{1}{n^p} = M_n$$

Now $\left|u_n(x)\right| \le M_n$ here $\Sigma M_n = \Sigma \frac{1}{n^p}$

by p series test the given series is convergent if p > 1

$\therefore$ $\Sigma u_n(x)$ is uniformly convergent on R.

3. Show that sequence $\{f_n(x)\}$ where $f_n(x) = nx(1-x)^n$ does not converge uniformly on [0,1].

Solution : Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}}$ $\left[\text{from} \frac{\infty}{\infty}\right]$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} = \lim_{n\to\infty} \frac{x(1-x)^n}{\log(1-x)}$$

$= 0$ $\left\{\because (1-x)^n \to 0 \text{ as } n \to \infty\right.$

Hence f(x) = 0 $\forall x \in [0,1]$

Now $M_n = \sup\left\{\left|f_n(x) - f(x)\right| = \sup\left\{nx(1-x)^n\right\}\right.$ $\left\{\text{taking } x = \frac{1}{n} \in [0,1)\right.$

$$\ge n\frac{1}{n}\left(1-\frac{1}{n}\right)^n$$

$$= \left(1-\frac{1}{n}\right)^n = \frac{1}{e} \quad \text{as } n \to \infty$$

Hence by M_n test $\{f_n(x)\}$ not converse uniformly.

4. Show that the series $\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ **converges in** $(1,\infty)$

Solution : Here $f_n(x)=\frac{1}{1+n^2x}$

$$|f_n(x)|=\left|\frac{1}{1+n^2x}\right|$$

$$\leq\frac{1}{1+n^2}<\frac{1}{n^2}=M_n\forall x\in[1,\infty)$$

$\sum_{n-1}^{\infty}M_n=\sum_{n-1}^{\infty}\frac{1}{n^2}$ is convergent

$\therefore$ By Weierstrass's M-test, the given series is uniformly convergent for all values of $x\in[1,\infty)$.

5. Show that the series $\sum_{n=1}^{\infty}\frac{a_nx^{2n}}{1+x^{2n}}$ **is uniformly convergent for all real x if** $\sum_{n=1}^{\infty}a_n$ **is absolutely convergent.**

Solution : Here $f_n(x)=\frac{a_nx^{2n}}{1+x^{2n}}$

since $\frac{x^{2n}}{1+x^{2n}}<1\forall x\in R$

$\therefore$ $|f_n(x)|=\left|\frac{a_nx^{2n}}{1+x^{2n}}\right|=|a_n|.\frac{x^{2n}}{1+x^{2n}}<|a_n|=M_n$

for all x $\in$ R

Since $\sum_{n=1}^{\infty} a_n$ is absolutly convergent, therefore $\sum_{n-1}^{\infty} M_n = \sum_{n-1}^{\infty} |a_n|$ is convergent.

$\therefore$ By Weierstrass's M-test, the given series is uniformly convergent for all real x.

6. Show that the series $\sum_{n=1}^{\infty} \frac{a_n x^n}{1+x^{2n}}$ is uniformly convergent for all real x if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Solution : Here $f_n(x) = \frac{a_n x^n}{1+x^{2n}}$

Let $$y = \frac{x^n}{1+x^{2n}}$$

Then $$\frac{dy}{dx} = \frac{(1+x^{2n})nx^{n-1} - x^n 2nx^{2n-1}}{(1+x^{2n})^2}$$

$$= \frac{nx^{n-1}(1+x^{2n} - 2x^{2n})}{(1+x^{2n})^2}$$

$$= \frac{nx^{n-1}(1-x^{2n})}{(1+x^{2n})^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = 0, x = 1$$

But $\frac{d^2y}{dx^2} < 0$ for x = 1. Hence y is maximum for x = 1

$$\max\left|f_n(x)\right| = \max\left|a_n y\right| = \left|\frac{a_n x^n}{1+x^{2n}}\right| = \left|\frac{a_n}{2}\right| \text{as } x = 1$$

$$= \frac{1}{2}\left|a_n\right| < \left|a_n\right| = M_n$$

Thus $\left|f_n(x)\right| < M_n \quad \forall x$

and $\Sigma M_n = \Sigma\left|a_n\right|$ is convergent as Σa_n is given to be absolutly convergent,

$\therefore$ By Weierstrass's M-test, the given series is uniformly convergent.

7. test the uniform convergence of the following, stating proper condition:

$$\sum_{n=0}^{\infty} a^n \cos nx$$

Solution : Write $u_n(x) = a^n \cos nx$.

Then $\left|u_n(x)\right| = \left|a^n \cos nx\right| \leq \left|a^n\right| = a^n$ if $a > 0$

Since $\sum a^n$ is convergent, for $0 < a < 1$, we have, by Weierstrass M-test, $\Sigma u_n(x)$ is uniformly convergent, if $0 < a < 1$.

Theorm 6. *Abel's test :* The series $\Sigma u_n(x) V_n(x)$ is uniformly convergent in [a, b] if

(i) $\Sigma u_n(x)$ is uniformly convergent in [a,b]

(ii) $\{V_n(x)\}$ is uniformly bounded in [a,b]

(iii) $\{V_n(x)\}$ is monotonic for each $x \in$ [a,b]

Proof : let $R_{n,p}(x)$ and $\gamma_{n,p}(x)$ denoted partial remainders of $\Sigma u_n v_n$ and Σu_n respectively so that

$$R_{n,p}(x) = u_{n+1}(x) v_{n+1}(x) + u_{n+2}(x) v_{n+2}(x) + \ldots + u_{n+p}(x) v_{n+p}(x)$$

and $\gamma_{n,p}(x) = u_{n+1}(x) + u_{n+2}(x) + ... + u_{n+p}(x)$

Then $\gamma_{n,1}(x) = u_{n+1}(x)$

$$\gamma_{n+2}(x) = u_{n+1}(x) + u_{n+2}(x)$$

$\therefore$ $u_{n+2}(x) = \gamma_{n2}(x) + \gamma_{n1}(x)$

Similarly

$$u_{n+3}(x) = \gamma_{n3}(x) - \gamma_{n2}(x) \text{etc}$$

by the uniform convergence of $\sum u_n(x)$ in $[a,b)$ for any given $\in > 0 \; \exists n_1 \in N$ s.t

$$n \ge n_1 \Rightarrow |\gamma_{n,p}| < \in$$

Also $\{v_n(x)\}$ is bounded uniformly in [a,b)

$\Rightarrow$ $\exists K > 0 \text{ s.t } |v_n(x)| < K \;\forall n, \forall x \in [a,b)$

Now $|v_n(x) - v_m(x)| \le |v_n(x)| + |v_m(x)| < K + K = 2K$

i.e. $|v_n(x) - v_m(x)| \le 2K \;\; \forall\, n, m \in N$

Now

$$R_{n,p}(x) = \gamma_{n1}(x) v_{n+1}(x) + [\gamma_{n2}(x) - \gamma_{n1}(x)] v_{n+2}(x)$$

$$+ [\gamma_{n3}(x) - \gamma_{n2}(x)] v_{n+3}(x) + ... + [\gamma_{np}(x) - \gamma_{n(p-1)}(x)] v_{n+p}(x)$$

or $R_{n,p}(x) = \gamma_{n.1}(x)[v_{n+1}(x) - v_{n+2}(x)] + \gamma_{n.2}(x)[v_{n+2}(x) - v_{n+3}(x)] + ...$

$$+ \gamma_{n.(p+1)}(x)[v_{n+p-1}(x) - v_{n+p}(x)] + \gamma_{np}(x) v_{n+p}(x)$$

For $n \ge n_1$

$$\left|R_{n,p}(x)\right| < \in \left[\left|v_{n+1} - v_{n+2}\right| + \left|v_{n+2} - v_{n+3}\right| + ... + \left|v_{n+p-1} - v_{n+p}\right| + \left|v_{n+p}\right|\right]$$

$\because$ $v_n(x)$ is monoatomic for $\forall x \in [a, b)$

So $\left|R_{n,p}(x)\right| < \in \left[v_{n+1} - v_{n+2} + v_{n+2} - v_{n+3} + ... + v_{n+p-1} - v_{n+p}\left| + \right|v_{n+p}\right|\Big]$

$$= \in \left[\left|v_{n+1} - v_{n+p}\right| + \left|v_{n+p}\right|\right] < \in (2K + K)$$

$\Rightarrow$ $\left|R_{n,p}(x)\right| < \in_1 \; \forall\, n \geq n_1 \quad \text{where } 3K \in = \in_1$

Hence $\Sigma u_n v_n$ is uniformly convergent in [a,b).

Theoram 7. (Dirichlet's test) : The series $\Sigma u_n(x) V_n(x)$ is uniformly convergent on [a, b] if

(i) The sequence Vn(x) is monoatonic decreasing sequence converging uniformly to zero for $x \in [a, b)$ and

(ii) $\{f_n(x)\}$ is bounded uniformly in [a,b) where $\sum_{n=1}^{n} u_n(x)$

Proof : Since $\{f_n(x)\}$ is uniformly bounded in [a,b] then $\forall n$ and $\forall x \in [a, b] \exists K$ s.t

$$\left|f_n(x)\right| < K$$

Now $f_{n+1} - f_n = \sum_{1}^{n+1} u_n - \sum_{1}^{n} u_n = u_{n+1}$

Similarly $f_{n+2} - f_{n+1} = u_{n+2}$

Since $\{v_n(x)\}$ converging uniformly to zero on [a,b] then $\forall$ any given $\in > 0 \exists n_0 \in M$

s.t $\left|v_n(x) - 0\right| < \in$ $\qquad \left\{\because \lim_{n\to\infty} v_n(x) = 0\right.$

We write

$$R_{n,p}(x) = u_{n+1}(x)v_{n+1}(x) + u_{n+2}(x)v_{n+2}(x) + ... + u_{n+p}(x)v_{n+p}(x)$$

Then $R_{n.p}(x) = [f_{n+1} - f_n]v_{n+1} + [f_{n+2} - f_{n+1}]v_{n+2} + ... + [f_{n+p} - f_{n+p-1}]v_{n+p}(x)$

Rearranging the term we get

$$R_{n.p}(x) = f_{n+1}(x)[v_{n+1} - v_{n+2}] + f_{n+2}[v_{n+2} - v_{n+3}] + ...$$

$$+ f_{n+p-1}[v_{n+p-1} - v_{n+p}] + f_{n+p}v_{n+p}$$

$$\therefore \quad R_{n.p}(x) < K\left[|v_{n+1} - v_{n+2}| + |v_{n+2} - v_{n+3}| + ... + |v_{n+p-1} - v_{n+p}| + |v_{n+p}|\right]$$

$$= K\left[|v_{n+1} - v_{n+2} + v_{n+2} - v_{n+3} + ... + v_{n+p-1} - v_{n+p}| + |v_{n+p}|\right]$$

Thus $|R_{n.p}(x)| < K\left[|v_{n+1} - v_{n+p}| + |v_{n+p}|\right]$

$$\leq K\left[|v_{n+1}| + 2|v_{n+p}|\right]$$

$$\leq K(\in + \in)$$

or $|R_{n.p}(x)| < \in_1 \quad \forall x \in [a.b)$ $\qquad \{\because \in_1 = 2\in$

Hence $\sum u_n(x)V_n(x)$ is uniformly convergent.

Some EXAMPLE on Abel's and Dirichlets test :

1. Test the series $\sum \frac{(-1)^{n-1}}{n} x^n$ for uniform convergence in [0,1].

Solutioin : Here $u_n(x) = \frac{(-1)^{n-1}}{n}, \quad v_n(x) = x^n$

(i) Clearly $\sum u_n(x)$ is uniformly convergent because it is convergent series of constant term.

(ii) Again $\{v_n(x)\} = \{x^n\}$ is uniformely bounded in [0,1] as $|v_n(x)| = x^n \le 1, \forall x \in [0,1]$

(iii) $\{v_n(x)\}$ is monotonic decreasing in [0,1], Hence by Abel's test $\Sigma \frac{(-1)^{n-1}}{n} x^n$ is uniformly convergent in [0,1]

2. If Σa_n is convergent series of positive constant prove that the series $\sum a_n x^n$ converges uniformly in [0,1]

Solutioin : Here $u_n(x) = a_n, \quad v_n(x) = x^n$

Then $\sum u_n(x) = \sum a_n$ is uniformly convergent in [0,1], because Σa_n convergent series of positive constant.

Now $\{v_n(x)\} = \{x^n\}$ is bounded in [0,1] for $|v_n(x)| = |x^n| \le v_x \text{ in } [0,1]$

$\therefore$ $\{v_n(x)\}$ is monotonic decreasing in [0,1],

3. Show that the series $\cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + ...$ converges uniformly in $0 < a \le x \le b \le 2\pi$

Solutioin : Here $u_n(x) = \cos nx, \quad v_n(x) = \frac{1}{n}$

$$f_n(x) = \sum_1^{\infty} \cos nx = \cos x + \cos 2x + ...+ \cos nx$$

$$= \frac{\cos\left[x + \left(\frac{n-1}{2}\right) x \sin\left(\frac{nx}{2}\right)\right]}{\sin \frac{x}{2}}$$

i.e. $\left|f_n(x)\right| \le \operatorname{cosec}\left(\frac{x}{2}\right) < K$

Since $\left\{v_n(x)\right\} = \left\{\frac{1}{n}\right\}$ is monotonic decreasing sequence converging uniformly to zero.

$\therefore$ $\{f_n(x)\}$ is bounded uniformly $\forall x \in [0, 2\pi]$

Hence by Dirichlet's test $\sum u_n v_n$ is uniformly convergent.

4. Show that series $\sum_{n=1}^{\infty}(-1)^{n-1}x^n$ converge uniformly in $0 \le x \le K < 1$

Solutioin : Here $u_n = (-1)^{n-1}, \quad v_n = x^n$

Since $f_n(x) = \sum_{n=1}^{\infty} u_n$ $\begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$

$\therefore$ $\{f_n(x)\}$ is bounded for all $n \in N$

Also $\{f_n(x)\}$ is positive monotonic decreasing sequence converging to zero. $\forall x \text{ in } 0 \le x \le K < 1$

Hence by Dirichlet's test the given series is uniformly convergent.

5.11. Uniform convergence and continuity

Suppose function $\{f_n(x)\}$ defined on [0,1]

Where $f_n(x) = x^n$ $\{0 \le x \le 1$

$\because$ $f_n(x) = x^n \quad \forall n \in N$ is polynomial function

$\forall n \in N$ f is defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x \le 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

Here the sequence $\{f_n(x)\}$ pointwise convergence to the function f which discontinuous at x = 1 on [0,1].

$\Rightarrow$ $\{f_n(x)\}$ is continuous function.

It is also true for the series of continuous function $\sum u_n(x)$ converging pointwise to f(x).

Theoram 8 : Let $\{f_n\}$ be the sequence of real valued function on [a,b] converges uniformly to the function f on [a,b] if each $f_n(x)$ is continuous on [a,b] then f is also continuous on [a,b].

Proof : Let t be arbitrary elements of [a,b] then we prove that f is continuous at t since each $f_n(x)$ is continuous on [a,b].

$\Rightarrow$ It is continuous at t since $\{f_n\}$ converges uniformly to f on [a,b]. Now for any given $\varepsilon > 0\ \exists\ m \in N$ s.t

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall\ n \geq m \quad \text{.......................................(1)}$$

In particular we get

$$|f_m(x) - f(x)| < \frac{\varepsilon}{3} \quad \text{...(2)}$$

and $$|f_m(t) - f(t)| < \frac{\varepsilon}{3} \quad \text{...(3)}$$

$\because$ f_m is continuous at t then $\exists$ $\delta > 0$ s.t

$$|f_m(x) - f_m(t)| < \frac{\varepsilon}{3} \text{ and } |x - t| < \delta \quad \text{..(4)}$$

If $|x - t| < \delta$ then we get

$$|f(x) - f(t)| = |f(x) - f_m(x) + f_m(x) - f_m(t) + f_m(t) - f(t)|$$

$$\leq \left|f(x) - f_m(x)\right| + \left|f_m(x) - f_m(t)\right| + \left|f_m(t) - f(t)\right|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \text{by (2), (3) and (4)}$$

Thus for given $\varepsilon > 0 \quad \exists\, \delta > 0$ s.t

$$\left|f(x) - f(t)\right| < \in \text{ whenever } \left|x - t\right| < \delta$$

f is continuous at t.

EXAMPLE

1. Show that series for which $f_n(x) = \dfrac{nx}{1+n^2x^2}, \quad 0 \leq x \leq 1$ **can't be differentiable term by term at x = 0.**

Solution : Here f(x) = 0 For 0 $\leq$ x $\leq$ 1

and $$f_n'(0) = \lim_{h\to 0} \frac{f_n(0+h) - f_n(0)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{n-h}{1+n^2h^2}}{h} - 0 = n = \infty \text{ at } n \to \infty$$

and $$f'(0) = 0$$

Hence $f'(0) \neq \lim\limits_{n\to\infty} f_n'(0)$

$\therefore$ The given series is not differentiable term by term at x = 0.

2. The givin series $\sum \cup_n(x)$ **for which**

$$f_n(x) = \frac{1}{2n^2}\log\left(1+n^4x^2\right)$$

Show that the series $\sum U_n(x)$ does not converges uniformly but it differentiable term by term.

Solution : Here $f'(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \log \frac{(1+n^4x^2)}{2n^2}$

$$= \lim_{n\to\infty} \frac{\frac{4n^3x^2}{1+n^4x^2}}{4n} = \lim_{n\to\infty} \frac{n^2x^2}{1+n^4x^2} = 0, \ 0 \le x \le 1$$

Hence $f'(x) = 0$

and $$= \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{xn^2}{1+n^4x^2} = 0, \ 0 \le x \le 1$$

$\therefore$ $f'(x) = \lim_{n\to\infty} f_n'(x)$

$\Rightarrow$ term by term differentiation hold the series $\sum U_n'(x)$ is not uniformly convergent in $0 \le x \le 1$.

$\therefore$ $\{f_n'(x)\}$ has 0 as a point of non uniform convergence.

3. Show that the function represented by $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ is differentiable for every x and its derivatives is $\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$

Solution : let $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ and $u_n(x) = \frac{\sin nx}{n^3}$

Then $U_n'(x) = \frac{\cos nx}{n^2}$

Thus $\sum_{n=1}^{\infty} u_n'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

Here $\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2} \quad \forall x$ and $\sum \frac{1}{n^2}$

is convergent by p-series test, p = 2 >1. Hence weierstrass's M-test, The series $\sum u_n'(x)$ is uniformly convergent for all x, It follows that series $\sum u_n(x)$ is differentiable term by term hence

$$f'(x) = \sum_{n=1}^{\infty} u_n'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

5.12. Uniform Convergence and Integration

Theoram 9. Let $\{f_n\}$ be a sequence of real valued function defined on closed and bounded interval [a,b] and let $f_n \in R[a,b]$ for n = 1, 2, 3, ...if $\{f_n\}$ converges uniformly to the function f on [a,b) then $f \in R[a,b]$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

Proof : Suppose for givin $\varepsilon > 0$

The sequence f_n converges uniformly to f on [a,b] then $\exists \; M > 0$ s.t $n \geq m$ and

$$x \in [a,b] \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \text{ ....(1)}$$

In particular for m = n

$$|f_m(x) - f(x)| < \frac{\varepsilon}{3(b-a)}$$

or $\quad f_m(x) - \frac{\varepsilon}{3(b-a)} < f(x) < f_m(x) + \frac{\varepsilon}{3(b-a)}$ ....(2)

$\because \quad f_m \in R[a,b] \; \exists$ partition

$p = \{ a = x_0, x_1, ...x_n = b\}$ of [a,b]

s.t $\quad U(p, f_m) - L(p, f_m) < \frac{\varepsilon}{3}$ ....(3)

Let $m_r(m), M_r(m)$, m_r, M_r denoted the infima and suprema of f_m and f on $[x_{r-1}, x_r]$ respectively. Now from (2) we get

$$f_m(x) < f(x) + \frac{\varepsilon}{3(b-a)} \quad \forall\ x \in [a, b)$$

$$\Rightarrow \quad m_r(m) \le m_r + \frac{\varepsilon}{3(b-a)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} m_r(m) \vartriangle x_r \le \sum_{r=1}^{n} m_r \vartriangle x_r + \frac{\varepsilon}{3(b-a)} \sum_{r=1}^{n} \vartriangle x_r$$

$$\Rightarrow \quad \angle(p, f_m) \le \angle(p, f) + \frac{\varepsilon}{3} \text{ ....(4)}$$

$$\Rightarrow \quad L(p, f_m) \le L(p, f)$$

similarly from (2) we get

$$f(x) < f_m(x) + \frac{\varepsilon}{3(b-a)}$$

$$\Rightarrow \quad M_\gamma < M_\gamma(m) + \frac{\in}{3(b-a)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} M_r \vartriangle x_r \le \sum_{r=1}^{n} M_r(m) \vartriangle x_r + \frac{\varepsilon}{3(b-a)} \sum \vartriangle x_r$$

$$\Rightarrow \quad U(p, f) \le U(p, f_m) < \frac{\varepsilon}{3} \text{ ....(5)}$$

Adding (4) and (5) we get

$$\cup(p,f)+L(p,f_m)\le L(p,f)+\cup(p,f_m)+\frac{2\varepsilon}{3}$$

$$\Rightarrow \quad \cup(p,f)-L(p,f)\le\cup(p,f_m)-L(p,f_m)+\frac{2\varepsilon}{3}$$

$$\le\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}=\varepsilon$$

$$\Rightarrow \quad f\in R[a,b]$$

Now for all m $n \geq m$

$$\left|\int_a^b f_n-\int_a^b f\right|=\left|\int_a^b (f_n-f)\right|\le\int_a^b |f_n-f|$$

$$<\frac{\varepsilon}{3(b-a)}\int_a^b dx \qquad \text{by}(1)$$

$$=\frac{\varepsilon}{3(b-a)}(b-a)=\frac{\varepsilon}{3}$$

Hence $\lim\limits_{n\to\infty}\left[\int_a^b f_n(x)dx\right]=\int_a^b f(x)dx$

5.13. Term by term integration

Theoram 10. Let $\sum\limits_{n=1}^{\infty}u_n(x)$ be the series of real value function defined on [a,b] s.t $u_n(x)\in R[a,b]$ for n = 1, 2, 3 if the series converges unifromly to f on [a,b) then

$$f\in R[a.b) \text{ and } \int_a^b\left[\sum_{n=1}^{\infty}u_n(x)\right]dx=\sum_{n=1}^{\infty}\int_a^b u_n(x)dx$$

Proof : let $f_n(x) = u_1(x) + u_2(x) + ...+ u_n(x)$

Sam of finite number of R- integrable function is equal to R- integrable.

$\Rightarrow \quad f_n \in R[a,b]$ for each n

Also for uniform convergence of the series $\sum u_n(x)$ is uniform convergence of the sequence f_n.

So that $f_n \to f$ uniformly on [a,b)

Hence $f \in R[a,b]$ by theoram 1

and $$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] dx = \int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx$$

$$= \lim_{n\to\infty}\int_a^b \left(\sum_{m=1}^{\infty} u_m(x)\right)dx$$

$$= \lim_{n\to\infty}\sum_{m=1}^{n}\int_a^b u_m(x)dx$$

$$= \sum_{m=1}^{\infty}\left[\int_a^b u_n(x)dx\right]$$

EXAMPLE

1. Show that $\int_0^1 \left(\sum_1^{\infty}\frac{x^n}{n^2}\right)dx = \sum_1^{\infty}\frac{1}{n^2(n+1)}$

Solution : By weierstrass M-test the series $\Sigma\frac{x^n}{n^2}$ is convergent unifromly for $0 \le x \le 1$ then it can be integrated term by term

$$\therefore \quad \int_0^1 \left(\sum_1^{\infty}\frac{x^n}{n^2}\right)dx = \sum_1^{\infty}\int_0^1\frac{x^n}{n^2}dx = \sum_1^{\infty}\left[\frac{x^{n+1}}{(n+1)x^n}\right]_0^1$$

$$= \sum_{1}^{\infty} \frac{1}{n^2 (n+1)}$$

2. Examine for term by term integration the series the sum of whose first n term is n²**x(1 - x)ⁿ 0 ≤ x ≤ 1.**

Solution : Here $f_n(x) = n^2 x(1-x)^n$

and $f(x) = \lim_{n\to\infty} f_n(x) = 0$ $\quad 0 \le x \le 1$

$$\therefore \quad \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2 x}{(1-x)^n} \qquad \left\{\text{from } \frac{\infty}{\infty}\right.$$

$$= \lim_{n\to\infty} \frac{2nx}{-(1-x)^{-n} \log(1-x)} \qquad \left\{\text{from } \frac{\infty}{\infty}\right.$$

$$= \lim_{n\to\infty} \frac{2x}{(1-x)^{-n} \left[\log(1-x)\right]^2} = 0$$

$$\Rightarrow \quad = \lim_{n\to\infty} f_n(x) = 0$$

But $\int_0^1 f_n(x)\,dx = \int_0^1 n^2 x(1-x)^n \,dx$

$$= \frac{n^2}{(n+1)(n+2)} = 1 \qquad \text{as } n \to \infty$$

So term by term integration is not valid in $0 \le x \le 1$

It follows that series is non uniformly convergent for $0 \le x \le 1$

It possible then for any given $\in > 0$

$$\left|f_n(x) - f(x)\right| = n^2 x(1-x)^n < 1$$

For $n \geq m \Rightarrow \left|f_n(x) - f(x)\right| = n^2 \frac{1}{n}\left(1 - \frac{1}{n}\right)^n = n\left(1 - \frac{1}{n}\right)^n$

$= \infty$ as $n \to \infty \quad 0 \leq x \leq 1$

Which is contradiction of (1) hence series is non unifromly convergent in $0 \leq x \leq 1$

3. Show that the series $f_n(x) = \frac{1}{1+nx}$ can be integrated term by term in 0 $\leq$ x $\leq$ 1 although they are not uniformly convergent in this interval.

Solution : Here $f_n(x) = \frac{1}{1+nx}$

and $f(x) = 0$ for $0 \leq x \leq 1$

$\therefore \quad \int_0^1 f(x)\,dx = 0$

and $\lim\limits_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim\limits_{n\to\infty}\int_0^1 \frac{1}{1+nx}\,dx$

$$= \lim_{n\to\infty}\frac{1}{n}\log(1+n) \qquad \left\{\text{from } \frac{\infty}{\infty}\right.$$

$$= \lim_{n\to\infty}\frac{\frac{1}{1+n}}{1} = 0$$

$\Rightarrow$ Series is term by term integrated but we have already seen zero is the point of non-uniform convergence of the series.

5.14. Uniform convergence and differentiation

Theoram 11. Let $\{f_n\}$ be a sequence of real valued function on [a,b] s.t

(i) f_n is differentiable on [a,b]

(ii) The sequence $\{f_n(c)\}$ convergence for some point c of [a,b].

(iii) The sequence $\{f'(x)\}$ converges uniformly on [a,b] then the sequence $\{f_n\}$ converges uniformly to differentiable limit f and

$$\lim_{n\to\infty} f'(x) = f'(x) \qquad \{a \le x \le b\}$$

Proof : Let $\in$ > 0 be given then by convergence of $\{f_n(t)\}$ and by uniform convergence of $\{f_n'\}$ on [a,b] $\exists$ positive integer m

s.t $$n \ge m,\ p \ge m \Rightarrow \left|f_n(c) - f_p(c)\right| < \frac{\varepsilon}{2} \text{.....................................(1)}$$

and $$\left|f_n'(x) - f_p'(x)\right| < \frac{\varepsilon}{2(b-a)} \qquad \{a \le t \le b\} \text{................(2)}$$

Now we apply mean value theorem of differential calculus to the function $f_n - f_p$ we have by (2)

$$\left|f_n(x) - f_p(x)\right| - \left[f_n(y) - f_p(y)\right] < \frac{\varepsilon(x-y)}{2(b-a)} \text{.........................(3)}$$

$$< \frac{\varepsilon}{2} \qquad \{\because |x-y| \le (b-a)$$

For all n, p $\ge$ m and all x $\in$ [a,b)

We show that for given ε > 0 $\exists$ m s.t

$$n \ge m, p \ge m, x \in [a,b] \Rightarrow \left|f_n(x) - f_p(x)\right| < \varepsilon$$

$\Rightarrow$ $\{f_n\}$ converge uniformly to function f

i.e. $$f(x) = \lim_{n\to\infty} f_n(x) \qquad \{a \le x \le b\}$$

Let us now fix a point x on [a,b] define

$$\phi_n(y) = \frac{f_n(y) - f_n(x)}{y - x}$$

and $\phi(y)=\dfrac{f(y)-f(x)}{y-x}$ ....(4)

For $a \le y \le b$ and $y \ne x$ then

$$\lim_{y\to x}\phi_n(y)=\lim_{y\to x}\frac{f_n(y)-f_n(x)}{(y-x)}=f_n'(x) \qquad \{n = 1, 2, 3\ldots \ldots\ (5)$$

Now for $n \ge m$, $p \ge m$ we get

$$|\phi_n(y)-\phi_p(y)|=\left|\frac{f_n(y)-f_n(x)+f_p(x)-f_p(x)}{y-x}\right|$$

$$\angle\frac{\varepsilon}{2(b-a)} \quad \text{by}\,(3)$$

$\Rightarrow$ $\{\phi_n\}$ converge uniformly for $y \ne x$

$\because$ $\{f_n\}$ converge to f

From equation (4)

$$\lim_{n\to\infty}\phi_n(y)=\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x}=\frac{f(y)-f(x)}{y-x}=\phi(y) \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

uniformly for $a \le y \le b$, $y \ne x$

Now apply theoram to $\{\phi_n\}$ (5) and (6) show that

$$\lim_{y\to x}\phi(y)=\lim_{n\to\infty}f_n'(x)$$

or $$\lim_{y\to x}\frac{f(y)-f(x)}{y-x}=\lim_{n\to\infty}f_n'(x)$$

or $$f'(x)=\lim_{n\to\infty}f_n'(x) \quad \forall x\in[a,b)$$

5.15. Term by term differentiation

Theoram 12. Let $\sum_{n=1}^{\infty} v_n(x)$ be the series of real valued differentiable function on [a,b] s.t $\sum_{n=1}^{\infty} v_n(c)$ converges for some point c in [a,b] and $\sum_{n=1}^{\infty} v'_n(x)$ converges uniformly on [a,b], Then series $\sum_{n=1}^{\infty} v_n(x)$ converges uniformly on [a,b] to a differentiable sum function f and

$$f'(x) = \lim_{n\to\infty} \sum_{r=1}^{n} v'_r(x) \qquad a \leq x \leq b$$

or if $a \leq x \leq b$ then

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} v_n(x)\right) = \sum_{n=1}^{\infty}\left[\frac{d}{dx} u_n(x)\right]$$

Proof : Suppose $f_n(x) = v_1(x) + v_2(x) + \ldots\ldots + v_n(x)$ then

$$f'_n(x) = v'_1(x) + v'_2(x) + \ldots\ldots + v'_n(x)$$

It follows the series $\sum_{n=1}^{\infty} v_n(x)$ and $\sum_{n=1}^{\infty} v'_n(x)$ are equivalant to the sequence $\{f_n\}$ and $\{f'_n\}$ respectively.

Now proof is same as the theoram (1)

Theoram 13. Let $\{f_n(x)\}$ be the sequence of real value function on [a,b] s.t

(i) $f_n(x)$ is differentiable on [a,b] for n = 1, 2, 3

(ii) The sequence $\{f_n(x)\}$ converge to f on [a,b]

(iii) The sequence $\{f'_n(x)\}$ conferges uniformly on [a,b] to g.

(iv) Each $f_n'(x)$ is continuous on [a,b]

Then $g(x) = f'(x)$

i.e $\lim_{n\to\infty} f_n'(x) = f'(x)$ $\quad a \le x \le b$

Proof : Here the sequence $\{f_n'(x)\}$ is uniformly convergent sequence of continuous function to g on [a,b] it follows by theoram 8 that g is continuous on [a,b) and also $\{f_n'(x)\}$ converges uniformly to g on [a,x] where x $\in$ [a,b] it follows by the theorm 9.

we get $\lim_{n\to\infty} \int_a^x f_n'(t) = \int_a^x g(t)\,dt$ ....(1)

By fundamental theoram of integral calculus we get

$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a)$ ....(2)

But by hypothesis

and $\left.\begin{array}{l} \lim_{n\to\infty} f_n(x) = f(x) \\ \lim_{n\to\infty} f_n(a) = f(a) \end{array}\right\rangle$ ....(3)

With the help of (1), (2) and (3) we get

$f(x) - f(a) = \int_a^x g(t)$ $\quad \{a \le x \le b$

$\Rightarrow \quad f'(x) = g(x)$ $\quad a \le x \le b$

or $\quad f'(x) = \lim_{n\to\infty} f_n'(x)$

Exercise : 4

1. Show that series $\Sigma \frac{(-1)^{n-1}}{(n + x^2)}$ uniform convergance for all value of x.

2. Show that series $\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...$ converges uniformly in $0 < a \le x \le b \le 2\pi$

3. Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} x^n$ converges uniformly in $0 < x \le K \le 1$.

4. Let $g_n(x) = \frac{1}{n} e^{-nx} (0 \le x < \infty)$. Prove that the sequence $\{g_n\}$ converges uniformly to 0 on $[0, \infty[$.

5. Show that 0 is a point of non-uniform convergence of the sequence $\{f_n(x)\}$ where

$f_n(x) = \tan^{-1} nx$ for $x \ge 0$.

6. Show that the series $(1-x)^2 + x(1-x)^2 + x^2(1-x)^2 + ...$ converges uniformly to $1 - x$ in $0 \le x \le 1$.

7. Let $f_n(x) = \frac{x^n}{1+x^n} (0 \le x \le 1)$. Show that $\{f_n\}$ converges uniformly on $[0, \frac{1}{2}]$

8. Let $f_n(x) = \frac{x}{n} e^{\frac{-x}{n}} (0 \le x \le \infty)$

   (i) Does $\{f_n\}$ converge uniformly to 0 in $[0, \infty[$?

   (ii) Does $\{f_n\}$ converge uniformly to 0 on $[0, 100[$?

9. Examine for term by term integration of the series for which

$f_n(x) = nxe^{-nx^2}$

10. Show that the series $f_n(x) = \frac{1}{1+nx}$ can be integrated term by term in $0 \le x \le 1$ although they are not uniformly convergent in this interval.

11. Examine for term by term integration the series $\Sigma x^{n-1}\left(1-2x^{n}\right)$ in the interval $0 \le x \le 1$.

12. Show that series $\Sigma \frac{1}{n^2+n^4x^2}$ is uniformly convergent for all real values of x and it can be differentiated term by term.

13. The given series $\sin x = x - \frac{x^3}{\angle 3} + \frac{x^5}{\angle 5} - \frac{x^7}{\angle 7} + ...$ show that one can be differentiation and obtain expansion of cos x,

    Show that the series $a^x = 1 + \frac{\log a}{\angle 1} + \frac{(\log a)^2}{\angle 2} + ... + \frac{(\log a)^{n-1}}{\angle n-1}$ can be integrated and differentiated term by term.

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